A COMPARATIVE STUDY OF LOWER BOUND BEARING CAPACITY SOLUTIONS

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CERTIFICATE

It is certified that the work contained in the thesis entitled **A COMPARATIVE STUDY OF LOWER BOUND BEARING CAPACITY SOLUTIONS** *by* SANJAY KUMAR SRIVASTAVA *has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.*

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ABSTRACT

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A COMPARATIVE STUDY OF LOWER BOUND BEARING CAPACITY SOLUTIONS

The primary object of this thesis is to compare the optimal lower bound bearing capacity solutions obtained by using the modified Lysmer's approach (Lysmer–Basudhar) with other solutions available in literature to asses its capability vis–a–vis other methods based on linear as well as non–linear programming techniques for isolation of the optimal stress field. This has been done with reference to bearing capacity of strip footings resting on the surface of a homogeneous soil deposit. Then by extending the Lysmer–Basudhar approach to bearing capacity of two layered soil deposits, similar study has been carried out and the obtained results have been compared with experimental observations and solutions based on method of characteristics and non–linear programming technique. The comparisons show that all the methods predict the bearing capacity reasonably well. For stratified deposits, Lysmer–Basudhar approach predicts values which are very close to experimental results.

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NOTATIONS

a_j	=	Coefficient to σ_j in linear function to be optimized.
a_{ij}	=	Coefficient to σ_j in linear constraint number i.
$[A]$	=	Coefficient matrix of the linear equality constraints.
b_i, b	=	Coefficients.
B	=	Width of footing.
$[B]$	=	9x7 matrix, geometrical property of the element.
C_u	=	Undrained cohesive strength of the soil.
C	=	Cohesion of the soil.
$\{\mathbf{D}\}$	=	Design vector.
$\mathbf{D_m}$	=	Optimum design vector.
δ	=	Angle of surface friction.
δ_t	=	Transition term between two types of penalty terms.
$F(\mathbf{D})$	=	Objective function.
$[G]$	=	9x7 matrix, geometrical property of the element.
$\{g\}$	=	9 component vector related to body forces inn_{th} element.
g_j	=	Inequality constraints.
H	=	Depth of the top layer of soil deposit.
$\{h\}$	=	9 component vector related to body forces inn_{th} element.
M	=	Total number of inequality constraints.
m, n	=	Element numbers.
r_k	=	Penalty parameter.
$[S]$	=	7x9 matrix, geometrical property of n_{th} element.
$\{\bar{s}_i\}$	=	3 component internal stress vector at node i of element n.
s	=	9 component stress vector which defines internal stresses in n_{th} element.
$[T]$	=	6x9 matrix, geometrical property of n_{th} element.

x_i	= x coordinate of nodal point i.
z_j	= z coordinate of nodal point j.
γ_x, γ_z	= Body forces per unit volume in x and z directions.
θ_{ij}	= Slope of element side connecting nodal points i and j.
μ, ζ, ϵ	= Known coefficients.
$\sigma_{x,i}$	= Normal stress on vertical plane through nodal point i.
$\sigma_{z,i}$	= Normal stress on horizontal plane through nodal point i.
σ^n	= Normal stress at nodal point i on plane parallel to element side jk of element n.
σ^n_{ij}	= Normal stress at point i of element n on side ij.
σ	= Stress vector which defines the complete stress field.
σ^n	= 7 component stress vector which defines external normal stress on n_{th} element.
$\tau_{zx,i}$	= Shear stress on horizontal plane at nodal point i.
$\tau_{i,j}$	= Shear stress at node i on the side connecting points i andj.
τ^n	= 6 component stress vector which defines the external shear stresses on n_{th} element.
ϕ	= Angle of internal friction.

CHAPTER 1

INTRODUCTION

1.1 General

In recent years lot of interest has been generated among the researchers in the area of geotechnical engineering to find the lower bound limit load for stability problems. The theoretical foundation of limit analysis has been under lain by Drucker, Greenberg and Prager (1952). For the materials with an associated flow rules, useful limit theorems (upper and lower bounds) can be applied to approximate the critical load, even if it cannot be determined exactly. An upper bound is often a good estimate of the collapse load but the lower bound is more important as it results in a safe design. However, except in few cases, it has not been possible to construct the statically admissible stress field for gravity loaded soil problems. To asses the state of the art a brief review pertaining to the work was carried out and is presented as follows.

1.2 Brief Review of Literature:

Since Coulomb first published his classical earth pressure theory in 1776, lot of development has taken place in soil plasticity over the years. However, only after 1952 when Drucker and Prager extended the study of Drucker et al. (1952) for perfectly plastic materials which obey Mohr–Coulomb yield criterion to granular material, limit analysis has extensively been used. Apart from limit analysis other methods that are commonly used to estimate the critical load of a foundation are limit equilibrium, method of characteristics and finite elements.

The review pertains mainly to the methods employed in determining the bearing capacity of footings resting on homogeneous and stratified soil deposits and leaves out many other details. For validation of the predictive models it is necessary to have experimental data and, as such, a few such work pertaining to bearing capacity have also been reviewed.

Button (1953), for the first time, analyzed the bearing capacity of continuous footing on two layered soil deposits using limiting equilibrium method. He has assumed general shear failure along the cylindrical slip surfaces starting at the edge of foundation and presented modified bearing capacity factors N_{cm} for saturated clays under undrained ($\phi_u = 0$) conditions with various values of C_2/C_1 ; ϕ_u is the angle of shearing resistance under undrained conditions and C_1 and C_2 are the undrained shear strengths of the top and bottom layer respectively. Two cases have been considered : (a) the shear strength, in each layer is constant with depth and (b) the shear strength of the upper layer decreases or increases with depth to a value C_1' and the lower layer has a constant strength C_1' with depth. Results have been presented in the form of design charts.

Tcheng (1957) has conducted tests for determining the bearing capacity of shallow foundations on a stratified soil deposit. The supporting soil consists of two layers with sand in upper layer of finite thickness and soft clay at the bottom being infinite. Experimental studies show that the mode of failure is punching along essentially vertical slip lines emanating from the foundation perimeter when the thickness of the top layer is less than 1.5 times the width of the footing. Based on model tests and theoretical

analysis of rupture surface observed, Tcheng has proposed empirical formulae for long rectangular footing resting on two layer soil system described above and has shown that the influence of the soft clay layer on bearing capacity becomes negligible when the sand layer thickness exceeds 3.5 times the footing width. The results obtained compare very well with the test results on the model.

Sokolovsky (1960, 1965) extensively used the method of characteristics to predict the lower bound bearing capacity and earth pressure problems; he also used the same method to study the stability of slopes. He considered homogeneous soils obeying Mohr–Coulomb failure criterion.

Yamaguchi (1963) has investigated the bearing capacity of a sandy layer of finite thickness resting on a soft clayey layer assuming a dispersion angle for pressure in sand layer below footing and taking uniform pressure at the top of the clay layer. He has presented expressions for bearing capacity values and has also discussed the method or principle to improve the ground economically. It has been shown that for small footings the top layer governs the bearing capacity values whereas for footings of large diameters clay layer is the controlling factor. He concludes that strip foundation is more economical when the sandy layer is firmer than clayey layer whereas raft foundation is preferable when the strength conditions are reversed. He also concludes that sand drain is better for improving two layered ground especially when sandy layer is in loose state.

Finn (1967) has presented a limiting plasticity theory on the basis of Mohr–Coulomb yield criterion and associated flow rule to provide upper and lower bound solutions to

problems in soil mechanics. It has been assumed that the soil reaches a perfectly plastic state and that no volume contraction occurs during plastic deformation. To illustrate the principles of the theory, it has been applied to classical problems of soil mechanics viz. critical height of vertical cut, pressures on retaining walls, ultimate bearing capacity of footings and bearing capacity of footings on slopes.

Using limiting equilibrium method, Siva Reddy and Srinivasan (1967) have further extended the work of Button (1953) to consider the non–homogeneity and anisotropy of soil with respect to shear strength. They have studied the effect of degree of anisotropy on bėaring capacity for both the cases considered by Button (1953). For values of degree of anisotropy $K > 1$, the ultimate bearing capacity is smaller than that for isotropic medium with constant vertical shear strength whereas for values $K < 1$ the ultimate bearing capacity is greater for anisotropic soils. The numerical results have been presented in the form of graphs for various degrees of anisotropy.

Davis (1968) has obtained lower and upper bound solutions under plane strain conditions for material with associated flow rule for ultimate bearing capacity of strip footing on pure cohesive soils, passive failure of cohesive–frictional soils and pressure on tunnel roofs overlain by clay by using discontinuous stress and velocity fields. By taking the problem of unconfined compression between rough end plates, it has been shown that the use of limit theorem is not justified for materials with non-associated flow rules.

Graham (1968) has investigated a numerical procedure based on the work of Sokolovsky (1960) to study the failure of retaining walls, slopes and deep strip footings and extended

it to take into account of non–homogeneity in the cross-section. The material has been assumed to be rigid plastic obeying Mohr–Coulomb failure criterion. The results obtained compare very well with the existing theories and test results.

Yokow et al. (1968) have extended Meyerhof's method (1951) to obtain the ultimate bearing capacity of a strip footing in two layered ground when the base of the footing is set in the supporting soil overlain by the weaker layer. The effect of shearing strength of the weak layer has been included in the bearing capacity analysis. The analysis is based on the assumption that soil is weightless rigid plastic body obeying Mohr–Coulomb yield criterion. Method of characteristics has been used to obtain the solutions. Two example problems have been undertaken to show the applicability of the proposed analysis.

Brown and Meyerhof (1969) have conducted experiments on the bearing capacity of layered clays using circular and strip footings for a range of layer thickness and clay strengths. Total stress analysis has been done. For stiff clays overlying soft clays failure occurs by punching of the footing through the top layer with full development of the bearing capacity of the lower layer. For the reverse case failure occurs mainly by squeezing of the top soft layer between footing and stiffer layer below, with more and more interaction between the layers as the strength ratio approached unity. They have presented their experimental results in the form of graphs for strip as well as circular footings resting on layered clays.

Mandel and Salencon (1969) have analyzed the bearing capacity of strip footing on two layered soil system using method of characteristics. The solution indicates that the

presence of a rigid layer below the bearing stratum results in an increase of bearing capacity.

Belytschko and Hodge (1970), using finite element technique, have presented an interesting general approach for finding the lower bound limit load for plane stress problems. Lower bounds have been obtained for a number of weakened slabs and compared with upper bounds obtained by previously available methods. A good agreement is noticed. The method is also of interest to geotechnical engineers due to its potential to be extended in solving stability problems.

Chen and Scawthorn (1970) have presented a critical discussion on the significance of the limit equilibrium and limit analysis solutions. They have shown that within the framework of idealizations the limit analysis approach is rigorous, competitive with limit equilibrium and in some instances much simpler. They have analyzed the bearing capacity of strip footings and the earth pressure problem using classical Coulomb plane failure mechanism and simple discontinuous stress field. Based on the results obtained, They have concluded that the assumption of perfect plasticity is very good for stability problems in soil mechanics.

Desai and Reese (1970) have used finite element method to investigate the behaviour of circular footings on a single as well as two layers of clay. The method employs non–linear stress–strain relationship, obtained from triaxial tests, to predict the load displacement relation of a steel footing. The results obtained for two layer soil system are found to be in good agreement with the test results.

Lysmer(1970), for the first time, developed a generalized method for lower bound analysis of plane problems in soil mechanics. The method uses simple three nodded triangular elements in which stress distribution has been assumed to be linear. Mohr–Coulomb yield criterion has been used. The problem has been formulated as a linear programming problem by linearizing the non–linear yield criterion. This has been applied to several earth pressure and bearing capacity problems. The results obtained compare very well with known solutions.

Krishnamurthy (1972) extended the method of characteristics to determine bearing capacity for layered C–ϕ soils obeying Mohr–Coulomb criterion of general shear failure. The values of cohesion, angle of internal friction and unit weight in each layer have been used to obtain stresses and slip lines. He has used finite difference technique to solve the differential equations in a manner similar to that of Sokolovsky's approach. Three different combinations of C and $\phi(a) C_2/C_1 = 2.0, \phi_2/\phi_1 = 0.75(b) C_2/C_1 = 4.0, \phi_2/\phi_1 = 0.6$ and (c) $C_2/C_1 = 0.4, \phi_2/\phi_1 = 1.25$ have been analyzed. He has also obtained the solutions for inclined loads on both homogeneous and layered soils. Results have been presented in the form of design charts.

Mandel and Salencon (1972) have obtained solutions for the bearing capacity of a soft ground layer overlying a rigid base using the theory of limiting equilibrium for plane strain conditions. Results have been obtained for the material obeying Coulomb's yield criterion for $0^0 \leq \phi \leq 40^0$. Effect of base friction and the ratio B/h(B=width of footing, h=depth of the top layer) have also been considered in the analysis and design

charts have been presented. It has been shown that for a perfectly rough contact the bearing capacity, starting from the classical value, increases steadily with B/h whereas for perfectly smooth contact the same decreases from the classical value, reaches a minimum and then in dealing with wide foundation it increases, becoming greater than the classical value.

Sabzevari and Ghahramani (1972) have presented an analytical study concerning the limit equilibrium of non–homogeneous soil medium satisfying non–linear yield criterion. Method of characteristic has been used in the analysis to derive the recurrence formulae. This has been applied to bearing capacity and earth pressure problems. The results obtained have been compared with those predicted by conventional theories of homogeneous soils. A significant difference between these two results show that the slip line fields as well as the stress distributions for bearing capacity and earth pressure problems in non-homogeneous soils with non-linear failure criterion, cannot be determined accurately from conventional limit equilibrium approach even if the analysis is based on the average values of cohesion, angle of internal friction and unit weight.

Chen and Davidson (1973) have obtained the upper bound limit load for both surface and embedded footings with smooth and rough bases. The soil is modeled as an elastic perfectly plastic material obeying Coulomb yield criterion. The analysis presented indicates that the significance of base friction is greatly reduced for deep footings. The results obtained compare well with existing solutions for both smooth and rough footings.

Davis and Booker (1973) have obtained upper bound solutions to problems of bearing capacity of clay which is inhomogeneous in vertical direction only. They have shown that the rate of increase of cohesion with depth plays the same role as density plays in the bearing capacity of homogeneous cohesive frictional soils. They have shown that for rigid footings, the bearing capacity depends upon the breadth and also that the roughness of footing may have small but significant effect in increasing the bearing capacity in contrast to the homogeneous case for which roughness has no effect. Results have been compared with those of slip circle analysis and it has been shown that the slip circle solutions may very seriously overestimate the bearing capacity of rigid footings.

Mayerhof (1974) has investigated the ultimate bearing capacity of both circular and strip footings resting on subsoils consisting of two layers for the case of dense sand on stiff clay and loose sand on stiff clay. The obtained results for different modes of soil failure have been compared with the results of model tests on circular and strip footings and some field observations of foundation failures. They have shown that the ultimate bearing capacity of footings on sand layer overlying clay can be expressed by punching shear coefficients for the case of dense sand on stiff clay and by modified bearing capacity coefficients for the case of loose sand on stiff clay. Theory and test results show that the influence of the sand layer thickness beneath the footing depends mainly on the bearing capacity ratio of the clay to sand, the friction angle of sand, the shape and the depth of the foundation.

Purushottamaraj et al. (1974) have presented upper bound limit analysis approach

for determining the ultimate bearing capacity of footings on two layered soils. They have considered the failure mechanism fundamentally similar to that of Prandtl–Terzaghi mechanism but with a different wedge angle. The critical wedge angles have been found in each case. However, they have presented bearing capacity charts for footings by varying only cohesion in layers and keeping the friction angle and unit weight constant.

Basudhar (1976) and Basudhar et al. (1979,1981) modified Lysmer's approach (1970) by incorporating the non–linear no–yield condition constraints directly in the analysis, thus formulating the lower bound optimization problem as non-linear programming problem. The constrained optimization problem has been converted to an unconstrained one using the extended penalty function method as suggested by Kavlie and Moe (1971). The sequential unconstrained minimization of the composite function so developed was carried out by using Powell's method along with quadratic interpolation technique for multidimensional and unidirectional search respectively (Fox, 1971; Rao, 1984). The method has been applied to bearing capacity and earth pressure problems. Results obtained compare very well with those of Lysmer's (1970).

Gioda and Donato (1979) have presented a numerical procedure based on finite elements and mathematical programming technique for the solution of geotechnical problems where elastic–plastic material behaviour is considered. The proposed approach can be adopted for geotechnical media characterized by any suitable yield condition, accounting, if necessary, for work hardening behaviour. Three geotechnical problems viz. determination of surface settlement produced by a strip load acting on a layered

soil deposit of finite thickness, horizontal and vertical displacement caused by an open excavation in a layered soil deposit and the surface settlements, linear deformation and stress states after the completion of shallow tunnel excavation have been dealt with to show the applicability of the proposed procedure. The results obtained have been compared with in-situ measurements and other available results. A reasonable agreement has been noticed.

Bottero et al. (1980) have presented an elasto-plastic finite element formulation using limit analysis theory to obtain lower and upper bounds of plane strain problems in soil mechanics. The problem has been formulated as a linear programming problem by using a linearized yield criterion for standard Tresca material with linear variation in stress and velocity fields. The problems of ultimate bearing capacity of strip footing, pull out capacity of foundations and slope stability have been dealt with to show the efficiency of the two proposed procedures.

Satyanarayana and Garg (1980) have proposed an empirical method to predict numerically the ultimate bearing capacity of footings on layered soils. They have given expressions for average values of shear strength parameters C and ϕ for the two layered system which can be used directly in the classical bearing capacity equations. The computed values are found to be in reasonable agreement with experimental results.

Hanna (1981) has conducted an experimental investigation to examine the validity of the method proposed by Satyanarayana and Garg (1980) for bearing capacity of strip and circular footings on two layered soils. He has concluded that more refinement and

further experimental and possibly field verifications are needed before recommending its implementation for practical purposes.

Kusakabe et al. (1981) have obtained the bearing capacity solutions of slopes loaded on top surface using upper bound theorem. The results have been compared with those obtained by conventional circular arc methods as well as by Kotter's stress characteristic equations. They have concluded that upper bound is useful from the engineering point of view because of the simplicity of the method. To check the validity of the upper bound solutions, model tests have also been conducted. The model tests show that the theory underestimates the bearing capacity. The failure mechanisms predicted by the theory with $\phi_u = 0$ assumption are in reasonable agreement with observation in model tests. Lysmer's (1970) method has also been used to obtain lower bound solutions to asses the validity of the upper bound analysis. The upper bound solutions are shown to be good approximation of exact solutions for bearing capacity of loaded slopes. The computed results are presented in the form of charts.

Caciaro and Cascini (1982) have proposed a mixed variational principle for the limit analysis of perfectly plastic continua in which the non-linear yield criterion and the associated flow rule appear through a 'penalty' function. Using mixed finite element discrete formulation and sequential unconstrained minimization technique, they have presented several numerical results for both structural mechanics and soil mechanics problems and have compared them with previously available exact and numerical solutions. A close agreement is noticed.

Hanna (1982) has investigated the ultimate bearing capacity of footings resting on subsoils consisting of a weak sand layer overlying a strong deposit. Based on model tests of strip and circular footings, he has shown that the bearing capacity of a weak sand layer overlying a strong deposit can be expressed by the classical equation of bearing capacity for homogeneous sand in conjunction with modified bearing capacity factors. The theory compares well with the available model test results. Design charts have been presented.

Baker and Frydman (1983) have studied the problem of finding the bearing capacity of a strip footing resting on the upper surface of a slope and have discussed the effect of non–linearity in the failure criterion of soil on the upper bound solution procedure. By considering the inherent non–linearity of the failure criterion, it has been shown that the upper bound solution procedure yields not only the minimum value of and the external load and the failure mechanism but also the stress distribution along the slip surface. They have demonstrated that there is a fundamental difference in the procedure used for applying the theorem to materials with linear and non-linear failure envelopes, which they have concluded to be due to the different roles played by the normality criterion in these two cases.

Baus and Wang (1983) have investigated, experimentally and analytically, the bearing capacity of footings located above a continuous void in silty clay soil. The analysis has been done by finite element method treating the soil as an elastic perfectly plastic material. Within the elastic range, the stress-strain relationship of the soil is described

by Hooke's law beyond which, the soil is as perfectly plastic in accordance with Von Mises yield criterion. It has been demonstrated that, for practical purposes, the void shape has negligible effect on the bearing capacity. Results also indicate an increase in bearing capacity with increasing depth of foundation when the depth of void is maintained constant. All the results have been presented in the graphical form.

Mizuno and Chen (1983) using finite element formulation and adopting Drucker–Prager models with associated as well as non–associated flow rules and cap models have obtained solutions for problems of flexible smooth and rough rigid footing resting on an over consolidated stratum of clay. They have observed that the velocity fields predicted by the plane cap model for both type of footing problems do not agree with that of the Prandtl's solution in the 'radial shearing zone' and 'near the surface zone', but, that predicted by Drucker–Prager and elliptic cap model agree well with Prandtl's solution for both the footing problems.

Reddy and Rao (1983) have obtained the upper bound bearing capacity of a strip footing on a two layer C–ϕ soil system exhibiting anisotropy and non–homogeneity in cohesion assuming Prandtl–Terzaghi failure mechanism with varying boundary wedge angles and presented the results in the form of non–dimensional charts. It is noticed that anisotropy and non–homogeneity in cohesion in each layer have considerable influence on the ultimate bearing capacity.

De Borst and Vermeer (1984) have examined the ability of a 15 nodded displacement type finite element to obtain the critical loads of soil structures for soils with high

frictional angle and with non–associated flow rules. Solutions have been presented for strip and circular footings, for the trap door problem and for the cone penetration test. With reference to footing problems, the accuracy of the numerical solution has been shown to be very high but stability problems occur when non–associated flow rules are applied.

Tamura et al. (1984) have investigated a numerical procedure to analyze the limit state of soil structures assuming the soil to be rigid plastic. The rigid plastic finite element method has been formulated on the basis of upper bound theorem. The numerical procedure has been investigated by typical problems viz. bearing capacity of shallow foundation and slope stability. Good agreement between the results and the existing solutions has been noticed.

Arai and Tagyo (1985) have developed a numerical procedure that furnishes a reasonable lower bound solution for the problems of bearing capacity and slope stability analysis. The stress field is discretized into quadrilateral elements and the formulated optimization problem is solved numerically using non–linear programming and sequential unconstrained minimization technique. It has been proved that the procedure provides an appropriate and stable lower bound solution for general soils which have cohesion, friction angle and its own weight, so far as the friction angle is not so large. However, the procedure cannot represent the arbitrary stress conditions at the boundary surface because a set of stresses is assumed to be constant within each element. Also, the procedure is difficult to apply for problems of soil structure interaction viz. earth

pressure problem since the procedure considers the stress as the independent variable and assumes the soil mass to be rigid perfectly plastic material. The validity of the procedure is successfully demonstrated through several case studies.

Tamura et al. (1987) have developed a rigid plastic finite element method for frictional materials. The stress–strain rate relation for a rigid plastic material of Drucker–Prager type under the assumption of associated flow rule has been derived. They have observed that materials with high friction angle values show somewhat unreasonable velocity field due to the dilatancy effect affecting the bearing capacity solutions. As such, a numerical technique for the non–associated flow rule to reduce such effects has been developed by satisfying both the yield condition and the normality for the plastic potential.

Sloan (1988) modified the method of Bottero et al. (1980) to obtain the lower bound solution for strip footing under plane strain conditions. A perfectly plastic soil model has been assumed, which may be either purely cohesive or cohesive frictional together with an associated flow rule. Mohr–Coulomb yield criterion has been assumed, the liner approximation of which enables the formulation to compute statically admissible stress field via finite elements and linear programming. Active set algorithm has been used to solve lower bound optimization problem which makes the method appreciably faster than than the displacement type of finite element method for predicting collapse load. He has solved bearing capacity problems of strip footing for homogeneous soil as well as for a purely cohesive soil which has increasing strength with depth. The obtained

solutions compare very well with the available Prandtl's and exact solutions.

Reddy et al. (1989), using the method of characteristics, obtained the bearing capacity factors for a circular footing placed at the interface of a two layered soil with the top layer being weaker than the bottom layer. The ground surface is taken to be horizontal up to a certain distance from footing beyond which it has been assumed to be inclined. The numerical results presented show that the bearing capacity factors are influenced by the stratification, strength of the soil layers and the depth at which the footing is placed and to a lesser extent by the other parameters.

Sloan (1989), assuming a perfectly plastic soil model which is either purely cohesive or cohesive frictional, has adopted the finite element formulation in conjunction with the upper bound limit theorem. It has been shown that the upper bound optimization problem may be solved efficiently by applying an active set algorithm to the dual linear programming problem. Upper bound solutions for strip footing as well as for a trapdoor in a purely cohesive soil have been obtained. These solutions compare very well with the available solutions for corresponding problems.

Reddy et al. (1990) used the method of characteristics to estimate the bearing capacity of strip footing placed at the interface of two layered soil with the bottom layer stronger than the top layer when the ground has an upward linear slope at a distance from the footing. The results presented show that the presence of an upward slope just adjacent to the footing and the presence of a stronger layer below the base of the footing increases the bearing capacity considerably.

Azam et. al. (1991) studied the performance of a strip footing on homogeneous and stratified soil deposits containing two soil layers both with and without a continuous void. They have used two dimensional finite element method and predicted the collapse load. To accommodate the non–linear stress strain characteristics of the foundation soil in the finite element analysis the incremental footing load is applied. For stratified soils, the obtained results compare well with the solutions of Vesic (1975).

Yong and Mohamed (1991) have developed an analytical method using FEM and non–linear stress analysis for predicting the performance of a muskeg deposit under loading. The deposit has been modeled as layered system consisting of three layers (surface mat, peat layer and mineral soil). The analytical results are found to be in good agreement with experimental results.

Chuang (1992) formulated the limit analysis of stability problems in geomechanics as a pair of primal-dual linear programs. The formulation provides a solution that is claimed to be both kinematically and statically admissible. For an assumed finite element mesh, the solution identifies the critical collapse mechanism among all the possible failure mechanisms contained within the given mesh and gives the corresponding values of both static and kinematic variables, together with the critical load parameter. Numerical solutions to bearing capacity problems as well as slope stability problems have been obtained. The method can readily handle failure surfaces of any arbitrary shape, external forces acting on the soil mass with varying pore water pressure, tension cracks filled with water and inhomogeneous material having both cohesion and angle

of internal friction. The computed results compare very well with the corresponding values of the analytical and numerical solutions.

Singh (1992) studied a number of stability problems in geotechnical engineering using Lysmer–Basudhar approach (Lysmer, 1970; Basudhar, 1976). For some of the problems he could compare his results with the lower bound solutions using method of characteristics, finite element and linear programming etc. The results obtained have been found to be in good agreement with the available results for most of the corresponding problems.

1.3 Motivation of the Work:

It can be seen from the reviewed literature that in nineteen hundred sixties and even thereafter, method of characteristics as suggested by Sokolovsky (1960, 1965) has been predominantly used by the research workers to predict the lower bound limit loads of stability problems in geotechnical engineering. But, in the early phases, solutions were available only for homogeneous and isotropic materials. In the early seventies efforts were made to extend these work to non–homogeneous and stratified deposits (Krishnamurthy, 1972; Purushottamaraj et al. 1974). The method of characteristics generally becomes very complicated for complex problems. As such, the need for a more generalized method to construct statically admissible stress field was felt by the research community. Thus a varity of methods combining the flexibility of finite element methods and the elegance of optimization technique in isolating the optimal lower bound limit load have been developed. In this direction the pioneering work of Lysmer

(1970), Bottero et al. (1980), Munro (1982) are worth mentioning. Another method which needs special attention is that of Hodge (1970); even though this method was developed for analyzing plate problems, the method is of interest to geotechnical engineers for the general nature of the solution procedure and its potentiality to solve stability problems. Apart from the developments of new methods to construct the statically admissible stress field, efforts have been continued over the last two decades to apply more and more sophisticated algorithms to enhance the computational efficiency of the original methods. With this in view Basudhar (1976) modified Lysmer's method and formulating the problem as a non–linear programming one, isolated the optimal stress field. However, the methods did not find much appreciation as these were constrained by the non–availability of high speed digital computers. In the eighties, with the revolutionary break through in the computers, new interest had been generated in applying these techniques. It is evident by the fact that after 1980 it was in 1985 that Arai and Tagyo made an effort to introduce quadrilateral elements and used non–linear programming technique to isolate the optimal stress field. In 1988 Sloan made an effort to improve upon Bottero et al's. approach (1980). Sloan and Asadi (1991) then used the same technique to a new class of problem namely the trapdoor problem. Side by side since 1989, research has been pursued at I.I.T.Kanpur, to apply Lysmer-Basudhar approach (Lysmer, 1970; Basudhar, 1976; Basudhar et al. 1979 and 1981) to different class of stability problems and the outcome has been reported by Singh (1992), Singh and Basudhar (1992, 1993*a* and 1993*b*).

So it is evident that various new methods of analysis are increasingly being suggested to predict the lower bound limit load for stability problems. As such, there is a need to asses the strength and weaknesses of these methods and make a comparative study. In addition there is also a need to extend these methods to new areas and, if possible, calibrate the models by comparing the obtained results with experimental values and also with other solutions available in the literature. With this in view an effort has been made in this thesis to make a comparative study of Lysmer–Basudhar approach, Arai and Tagyo approach, Bottero–Sloan's approach and Munro–Chuang approach. In addition Lysmer–Basudhar approach has further been extended to find the bearing capacity of strip footing resting on the surface of two layered soil deposits.

The comparison of the predicted results using Lysmer-Basudhar approach with both experimental and theoretical values reported in literature enables one to judge the capability of the method used in the present study vis a vis other methods.

1.4 Scope and Organization:

In chapter 2 the original formulation of the Lysmer–Basudhar approach adopted in this thesis for analyzing the stability problems has been presented in brief.

In chapter 3 a comparative study of the different techniques has been undertaken and presented with reference to a smooth strip surface surface footing resting on homogeneous, general C–ϕ soil.

In chapter 4 Lysmer–Basudhar approach has been extended to find out the bearing capacity of both rough and smooth surface strip footings resting on a two layered soil

deposit. A study regarding the extensibility of the stress field has been presented. The obtained results have been compared with available experimental and numerical solutions reported in the literature.

Generalized conclusions and scope of future work has been presented in chapter 5.

In the appendix, a mathematical proof, as given by Lysmer (1993), of the fact that satisfying the no–yield condition only at the nodes of the triangular elements is sufficient to ensure that yielding does not occur at any point in the element, has been presented.

CHAPTER 2

GENERAL METHOD OF ANALYSIS

2.1 General:

The generalized method of lower bound limit analysis as developed by Lysmer (1970) and subsequently modified by Basudhar (1976) to incorporate the non-linear no-yield condition constraints directly in the analysis has been adopted for analyzing the problems.

The method generates stress fields which are in equilibrium everywhere and do not violate the Mohr-Coulomb failure criterion at any point inside the soil medium. Furthermore since infinitely many stress fields satisfy these conditions for any given problem, the method is formulated as a mathematical programming problem to isolate stress fields which yield high lower bounds. The stress field that is considered in this method has the property that all stresses vary linearly within each element of some mesh which cover the soil mass under study. For the sake of completeness the method is presented herein in brief.

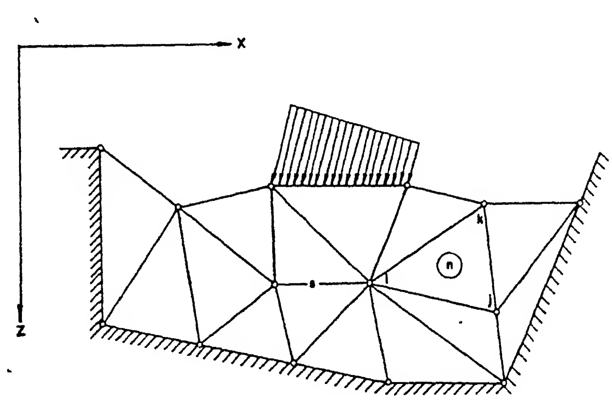

Fig 2.1 Discretization of the soil mass for a typical problem

The first step in the analysis of a typical problem, such as the bearing capacity problem shown in Fig 2.1, is the disretization of the soil mass under consideration into a mesh of finite number of triangular elements. If possible, the zone of influence considered for discretization should be based on previous experimental and theoretical studies. Further discretization of this zone should be done keeping in mind the guide lines suggested by Lysmer (1970). All nodal points, elements and element sides are then numbered in some arbitrary order. It can be shown that a mesh consisting of p elements connecting at q nodal points will have $p + q - 1$ element sides.

2.2 Element Equilibrium:

The geometry of a typical element n, and the external stresses and the body forces acting on this element are shown in Fig 2.2.

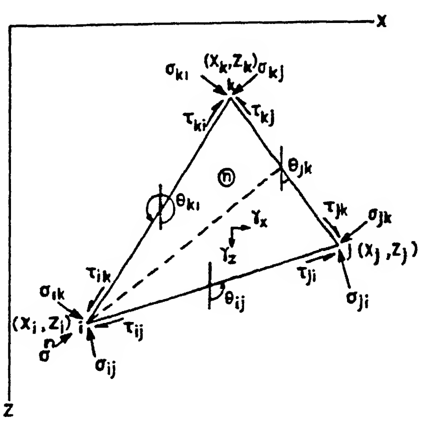

Fig 2.2 Definition sketch and body forces for n^th element

The stresses are assumed to vary linearly within each element, hence the stresses only at the nodes are considered. In addition, one internal stress σ^n is defined as the normal stress at node i acting on a plane parallel to the side jk. The normal stresses shown in Fig 2.2 are collected into a 7-component stress vector $\{\sigma^n\}$ defined by

$$[\{\sigma\}^n]^T = \{\sigma^n \sigma_{ik} \sigma_{ij} \sigma_{ji} \sigma_{jk} \sigma_{kj} \sigma_{ki}\} \tag{1}$$

and the external shear stresses are collected into a 6- component stress vector $\{\tau^n\}$ defined by

$$[\{\tau\}^n]^T = \{\tau_{ik} \tau_{ij} \tau_{ji} \tau_{jk} \tau_{kj} \tau_{ki}\} \tag{2}$$

the internal stresses in each element are collected into 9-component stress vector $\{s\}$ defined as

$$\{s\}^T = \{\bar{s}_i \bar{s}_j \bar{s}_k\} \tag{3}$$

$$\text{with, } \{s_i\}^T = \{\sigma_{z,i} \sigma_{x,i} \tau_{zx,i}\} \text{ etc.} \tag{4}$$

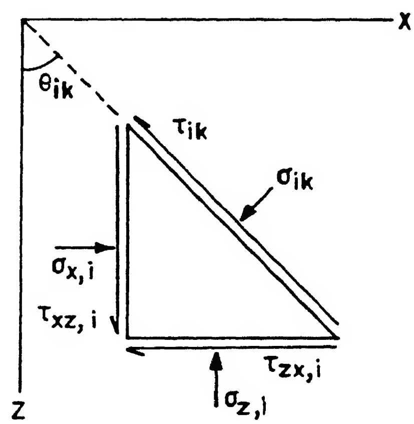

Fig. 2.3 Internal stresses at point i.

where, $\{\bar{s}_i\}$ are the internal stresses at node i. The equilibrium conditions for the infinitesimal triangle at node i shown in Fig.2.3, are expressed in terms of σ_{ik} and τ_{ik}

as:

$$\sigma_{ik} = \sigma_{z,i} \sin^2\theta_{ik} + \sigma_{x,i} \cos^2\theta_{ik} - \tau_{zx,i} \sin 2\theta_{ik} \tag{5}$$

$$\tau ik = 0.5(\sigma_{x,i} - \sigma_{z,i}) \sin 2\theta_{ik} + \tau zx, i \cos 2\theta_{ik} \tag{6}$$

Similar equations are written for nodes j and k and substituted in eqns. (1) and (2) to yield

$$\{\sigma\}^n = [s] \tag{7}$$

$$\{\tau\}^n = [T] \tag{8}$$

Matrices $[S]$ and $[T]$ are the geometric properties of nth element. The conditions of internal equilibrium are

$$\begin{aligned} \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{zx}}{\partial x} &= \gamma_z \\ \frac{\partial \tau_{zx}}{\partial z} + \frac{\partial \sigma_x}{\partial x} &= \gamma_x \end{aligned} \tag{9}$$

To satisfy these relations, the linear stress fields within the nth element are expressed in the following form:

$$\begin{aligned} \sigma_z &= c_1 z + c_2 x + c_3 + \gamma_z z \\ \sigma_x &= c_4 z + c_5 x + c_6 + \gamma_x x \\ \tau_{zx} &= -c_5 z - c_1 x + c_7 \end{aligned} \tag{10}$$

Thus the stress field depends on seven parameters c_i which may be combined into the vector

$$\{c\}^T = \{c_1\ c_2\ c_3\ c_4\ c_5\ c_6\ c_7\} \tag{11}$$

Using eqns. (3), (4) and (10) the stress vector $\{s\}$ can be written as:

$$\{s\} = [G]\{c\} + \{g\} \tag{12}$$

where

$$\{g\}^T = \{\gamma_z z_i,\ \gamma_x x_i,\ 0,\ \gamma_z z_j,\ \gamma_x x_j,\ 0,\ \gamma_z z_k,\ \gamma_x x_k,\ 0\} \tag{13}$$

is purely a function of geometric properties of nth element. Using equns. (7) and (12),

$$\{c\} = ([S][G])^{-1}\{\sigma\}^n - ([S][G])^{-1}[S]\{g\} \tag{14}$$

which when substituted into eqn. (12) gives

$$\{s\} = [B]\{\sigma\}^n + \{h\} \tag{15}$$

$$\text{where } [B] = [G]([S][G])^{-1} \text{ and } \{h\} = \{g\} - [B][S]\{g\} \tag{16}$$

Both $[B]$ and $\{h\}$ are geometric properties of nth element. Using eqns. (8) and (15), we get

$$\{\tau\}^n = [T][B]\{\sigma\}^n + [T]\{h\} \tag{17}$$

which is the equilibrium condition for nth element.

2.3 Interface Equilibrium:

The elements of $\{\sigma\}^n$ vectors for all the elements are collected into a general $\{\sigma\}$ vector. A system consisting of p elements connected at q nodal points will have $(3p + 2q - 2)$ stress variables in $\{\sigma\}$ vector, which are the principal unknowns.

The continuity of normal and shear stresses across any interface as shown in Fig. 2.4 requires:

$$\sigma_{ij}^m = \sigma_{ij}^n \qquad and \qquad \tau_{ij}^m = \tau_{ij}^n \tag{18}$$

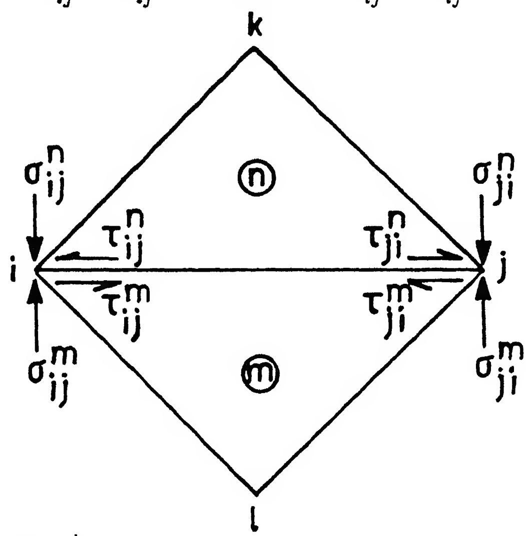

Fig 2.4 Continuity of nodal stresses

for all corresponding values of i, j, m and n. These conditions yield a set of linear equality constraints in terms of the principal unknowns. The total number of interface equilibrium condition is equal to twice the number of element sides in contact.

2.4 External Boundary Conditions:

The boundary stresses on the external faces of the system may be expressed either as :

$$\tau_{ij} \leq \mu \sigma_{ij}$$

$$\sigma_{ij} = \zeta \qquad and/or \qquad \tau_{ij} = \epsilon \sigma_{ij} \tag{19}$$

Equns. (18), and (19) can be transformed into the form :

$$\sum_{j=1}^{(3p+2q-2)} a_{ij}\sigma_j = b_i \tag{20}$$

and/or

$$\sum_{j=1}^{(3p+2q-2)} a_{ij}\sigma_j \leq b_i \tag{21}$$

2.5 No-yield condition:

For static admissibility, the stress field should not violate the Mohr-Coulomb yield criterion at any point in the soil medium. Since all stresses are assumed to vary linearly within each element, it is sufficient to satisfy the no yield condition at the element corners only. The condition at the node i can be written as:

$$(\sigma_{z,i} - \sigma_{x,i})^2 + (2\tau_{zx,i})^2 \leq [(\sigma_{z,i} + \sigma_{x,i})\text{Sin}\phi + 2C\ \text{Cos}\phi]^2 \tag{22}$$

Eqn. (22) is expressed in terms of the principal unknowns as follows:

$$\sigma_{z,i} = Z_i\{s\}, \sigma_{x,i} = X_i\{s\} \text{ and } \tau_{zx,i} = T_i\{s\} \tag{23}$$

where,

$$\begin{aligned}
Z_i &= (1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
X_i &= (0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)\\
T_i &= (0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0)
\end{aligned} \tag{24}$$

similarly for nodes j and k ,

$$\begin{aligned}
Z_j &= (0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0)\\
X_j &= (0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0)\\
T_j &= (0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0)
\end{aligned} \tag{25}$$

and

$$\begin{aligned}
Z_k &= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0)\\
X_k &= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0)\\
T_k &= (0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 1)
\end{aligned} \tag{26}$$

Equns. (21) and (22) yield:

$$(A_i\{s\})^2 + (2T_i\{s\})^2 - (B_i\{s\}\ \mathrm{Sin}\phi + 2C\ \mathrm{Cos}\phi)^2 \leq 0 \tag{27}$$

where,

$$A_i = \ (Z_i - X - i)$$

$$B_i = \quad (Z_i + X - i) \tag{28}$$

Now, equn. (27) can be rewritten in terms of $\{\sigma\}^n$ as:

$$[A_i\,([B]\sigma^n + h)]^2 + [2T_i\,([B]\sigma^n + h)]^2$$

$$-[B_i\,([B]\sigma^n + h)\sin\phi + 2C\cos\phi]^2 \leq 0 \tag{29}$$

Similar relations can be obtained for the nodes j and k. The elements of $\{\sigma\}^n$ vector can be picked up from the general stress vector $\{\sigma\}$. The total number of non-linear equality constraints will be 3p. It is sufficient to satisfy the non–linear equality constraints only at the nodal points of the triangular elements to ensure that there is no yield at any point within the element. The proof of the same as provided by Lysmer (1993) is given in the appendix.

2.6 Objective function, Design Variables, Design Restrictions and Reduction of Design Variables:

Since in general infinitely many stress fields will satisfy the aforementioned condition of static admissibility. the isolation of the stress field which optimizes the objective function is important. In almost all the problems, the stress quantity is a linear combination of surface stresses σ_{ij} and τ_{ij} Using eqn. (17) this quantity can be transformed into a linear combination of principal unknown σ_j which are termed as design variables. The problem can be stated as:

$$\text{OPTIMIZE} \sum_j a_j \sigma_j \tag{30}$$

The design restrictions are interface equilibrium and the external boundary conditions. As soil cannot take tension, the following constraints are also introduced,

$$-\sigma_j \leq 0 \tag{31}$$

eqns. (20) and (28) are presented in general term as:

$$g_j \leq 0 \tag{32}$$

The equality constraints (Eqn. 20) can be rewritten in matrix notation as

$$[\mathbf{A}]\{\sigma\} = \{b\} \tag{33}$$

Some of the elements of $\{\sigma\}$ vector are specified at the boundary. The following relation :

$$[\mathbf{A}^*]\{\sigma^*\} = \{b^*\} \tag{34}$$

can be arrived at by eliminating the columns of [A] matrix corresponding to the known elements of the $\{\sigma\}$ vector. $\{\sigma^*\}$ is a vector which is achieved by eliminating the known elements of $\{\sigma\}$ vector. $\{b^*\}$ vector is calculated as follows:

$$\{b^*\} = \{b\} - [\mathbf{A}']\{\sigma'\} \tag{35}$$

$[\mathbf{A}']$ matrix contains the columns that are removed from [A] matrix and $\{\sigma'\}$ contains those elements of $\{\sigma\}$ vector that are specified.

The following steps are performed for the general rectangular matrix $[\mathbf{A}^*]$:

Stepl. The rank and the linearly dependent rows and columns if there be any of the given matrix, are determined.

Step2. A sub matrix of maximal rank is expressed as product of triangular factors.

Step3. The non basic rows are expressed in terms of the basic ones.

Step4. The basic variables are expressed in terms of the free variables.

By considering these free variables as design variables and expressing the remaining basic variables in terms of these design variables the equality constraints (Eqn. 33) are implicitly satisfied. Such a technique helps in reducing the complexity of the problem by eliminating the equality constraints and there by reducing the dimensionality of the problem. The independent design variables so obtained are collected in $\mathbf{D}$ vector.

The rank(r) is determined using the standard Gaussian elimin- ation technique with complete pivoting. This implies that the rows and columns of the given $m'xn'$ matrix $[\mathbf{A}^*]$ are interchanged at each elimination step if necessary. In general the following cases may arise:

1. $r = m' = n'$

$[\mathbf{A}^*]$ is non-singular and $[\mathbf{A}^*]\{\sigma^*\} = \{b^*\}$ has uniquely determined solution.

2. $r < m'$

$[\mathbf{A}^*]$ is not row regular and the solution of Eqn.(33) exists only if the remaining $(m' - r)$ equations are linearly dependent.

3. $r < n'$

$[\mathbf{A}^*]$ is not column regular and the system has no trivial solution.

Cases (1) and (2) may occur combined. The solution if it exits, can be uniquely determined if $r = n'$, otherwise, it contains $(n' - r)$ free parameters.

The basic variables (σ^{**}) are expressed in terms of the free design variable $(\mathbf{D})$ as follows :

Once the steps 1,2 and 3 of the enunciated reduction process are carried out the Eqn. (34) is reduced to a form,

$$[\mathbf{A}^*]^r\{\sigma^*\}^r = \{b^*\}^r \tag{36}$$

where, the superscript denotes the rth elimination step and

$$[\mathbf{A}^*]^r = \begin{pmatrix} \mathbf{L} \\ \\ \mathbf{LR} \end{pmatrix} (\mathbf{U}, \mathbf{UR}) \tag{37}$$

where, L is a unit lower triangular matrix of dimension rx r.

U is a unit upper triangular matrix of dimension rx r.

LR is of dimension $(m' - r)xr$; if the matrix $[A^*]$ is row regular

that is $(m' = r)$, **LR** is absent in the final factorization.

UR is of dimension $rx(n' - r)$; if the matrix $[\mathbf{A}^*]$ is column regular that is $(n' = r)$, **UR** is absent in the final factorization.

Let $\{\sigma^*\}^r$ and $\{b^*\}^r$ be partitioned into

$$\begin{Bmatrix} \sigma^{**} \\ \mathbf{D} \end{Bmatrix} \quad \textit{and} \quad \begin{Bmatrix} b_1^* \\ b_2^* \end{Bmatrix}$$

Then for a consistent system of equations

$$\sigma^{**} = \mathbf{U}^{-1}\mathbf{L}^{-1}b^* + \mathbf{HD} \tag{38}$$

$$\text{where,} \qquad \mathbf{H} = -\mathbf{U}^{-1}\mathbf{UR} \tag{39}$$

$$\text{and} \qquad b_1^* = \mathbf{LU}\sigma^{**} + \mathbf{LURD} \tag{40}$$

$$b_2^* = \mathbf{LRL}^{-1}b_1^* \tag{41}$$

In the present thesis subroutine MFGR developed by IBM has been used to perform the calculations enunciated in steps 1 to 4 of the reduction process.

2.7 Mathematical Programming Problem:

Determination of the minimum value of the objective function subject to the inequality constraints as described above is formulated as mathematical programming problem which is stated as follows:

Find $\mathbf{D_m}$ such that

$$F(\mathbf{D_m}) = \Sigma \mathrm{a_j}\sigma_\mathrm{j} \qquad \text{is minimum} \tag{42}$$

subject to $g_j(\mathbf{D_m}) \leq \mathbf{0}$

There is no loss of generality even though the problem is cast as a minimization problem as maximum of a function can be achieved by minimizing the negative of the function.

The constrained problem is converted into an unconstrained optimization problem with the help of extended penalty function technique as suggested by Kavlie and Moe (1971). The Sequential Unconstrained Minimization of the developed composite function is carried out using Powell's conjugate direction algorithm (Powell, 1964) along with Quadratic interpolation technique for linear minimization to isolate the optimal solution. These methods are available in any standard text book on Optimization (Fox, 1971; Rao, 1984). The composite function $\phi(\mathbf{D}, r_k)$ is developed by blending the objective function and constraints as follows:

$$\phi(\mathbf{D}, r_k) = F(\mathbf{D}) + r_k \sum_{j=1}^{M} \mathrm{G}[g_j(\mathbf{D})] \tag{43}$$

The function $G[g_j(\mathbf{D})]$ is chosen as:

$$G[g_j(\mathbf{D})] = \begin{cases} 1/g_j(\mathbf{D}) & g_j(\mathbf{D}) \leq 0 \\ 2\lambda - g_j(\mathbf{D})/\lambda^2 & g_j(\mathbf{D}) \geq \lambda \end{cases} \tag{43}$$

where $\lambda = -r_k/\delta_t$

and δ_t = a constant that defines the transition between the two types of penalty terms.

In this approach infeasible starting points are readily acceptable to the minimization algorithm, which makes it a powerful technique for solving various engineering problems even if an initial feasible design vector is difficult to guess.

CHAPTER 3

LOWER BOUND BEARING CAPACITY OF SURFACE STRIP FOOTINGS IN HOMOGENEOUS SOILS

3.1 Introduction

The ultimate bearing capacity of strip footings resting on homogeneous soils has been widely studied by several investigators. The methods of analysis employed are based on limit equilibrium, limit analysis and finite element techniques. Limit analysis solutions provide either a lower or an upper bound to the critical load. Chapter 1 of the thesis gives a brief account of the available lower and upper bound bearing capacity solutions. It is seen that for isolating the optimal stress field, two approaches viz. linear programming approach (Lysmer, 1970; Bottero et al., 1980; Munro, 1982; Sloan, 1988; Chuang, 1992) and non–linear programming approach (Basudhar, 1976; Basudhar et al, 1979, 1981; Arai and Tagyo, 1985) have been employed. But no comparative study has been taken up to establish the relative merits and demerits of linear and non–linear programming approaches of finding the optimal lower bound bearing capacity solutions.

As such, a study has been under taken and presented with reference to a smooth strip footing resting on the surface of homogeneous soils using Lysmer–Basudhar approach. The obtained solutions are then compared with the values which had been computed by using linear programming and reported in the literature.

3.2 Footing on Homogeneous $C - \phi$ soil

3.2.1 The Problem

Fig. 3.1 shows a smooth strip footing resting on the surface of a general cohesive-frictional ($C - \phi$) soil. The objective is to determine the bearing capacity factor N_q when the shear strength parameters C and ϕ are 1.00 kPa and 40^0 respectively.

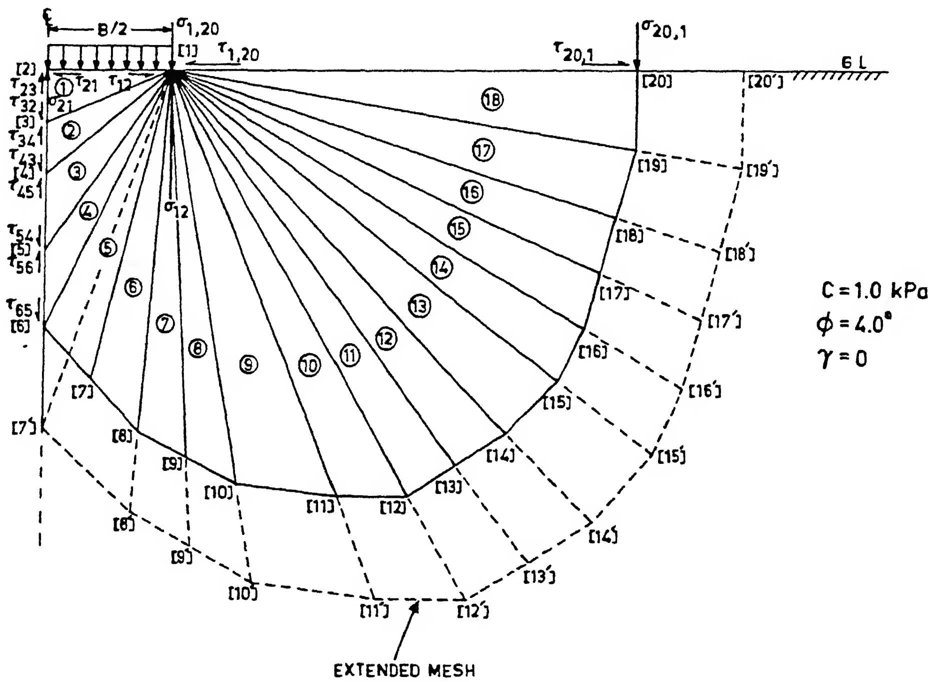

Fig 3.1 Strip footing on homogeneous $C - \phi$ soil

The exact collapse pressure for a smooth strip footing resting on the surface of a cohesive-frictional weightless soil may be written as

$$q_f = CN_c + qN_q$$

where

$$N_q = \exp(\pi \tan \phi) \tan^2(\pi/4 + \phi/2),$$

$$N_c = (N_q - 1)\cot\phi$$

and q is the overburden pressure. From the above relations N_q can be obtained as follows:

$$N_q = \frac{q_f}{C\cot\phi} + 1$$

For $C = 1.00$ kPa and $\phi = 40^0$, Prandtl's solution for N_q is 64.20; this value has been reported to be exact by Sloan (1988). This problem was previously solved by Sloan as a bench mark problem to demonstrate the effectiveness of his technique. The same is adopted to make a detailed comparative study of the Lysmer-Basudhar approach with Bottero–Sloan's approach.

3.2.2 The Objective Function

The objective function is $-(\sigma_{12} + \sigma_{21})$. Bearing capacity q_f is equal to half of the absolute value of the objective function.

3.2.3 The Boundary Conditions

The boundary conditions for the meshes shown in Fig. 3.1 are

$$\sigma_{12} = \sigma_{21} = 0$$

and $\tau_{12} = \tau_{21} = \tau_{23} = \tau_{32} = \tau_{34} = \tau_{43} = \tau_{45} = \tau_{54} = \tau_{56} = \tau_{65} = \tau_{1,20} = \tau_{20,1} = 0$

3.2.4 Results and Discussions

Results were obtained on CONVEX C–220 computer system for different number of elements (6, 12, 18, 24, 36 and 48) and a convergence study was made to determine the optimum number of elements. However, only the mesh with eighteen elements is shown

in Fig. 3.1 and the others are not shown for the sake of space and brevity. The mesh pattern was chosen keeping in mind that the singular points should be common to as many triangular elements as possible. In Fig. 3.1, point 1 is such a point and all the eighteen elements have this as a common point. The obtained bearing capacity factors N_q, total number of principal unknowns, design variables, equality constraints, inequality constraints and the total number of function evaluations to achieve the optimal objective function value are presented in Table 3.1.

Table 3.1 Optimization Details for the Bearing Capacity Problem

No. of elements	6	12	18	24	36	48
N_q	47.84	60.70	62.42	62.65	62.56	61.91
No. of Principal unknowns	32	62	92	122	182	242
No. of Design variables	13	31	45	63	91	127
No. of Equality constraints	18	30	46	58	90	114
No. of Inequality constraints	48	96	144	192	288	384
No. of function evaluations	1023	23898	76459	123530	226625	416725

The magnitude of problems that are likely to be faced may be imagined from the data presented in Table 3.1. As the number of elements is increased from six to forty eight, almost tenfold increase in the number of design variables occured; for meshes with six and fortyeight elements the number of design variables are 13 and 127 respectively. In literature (Fox, 1971) it is generally suggested that for problems with more than fifty design variables, variable matric method should generally be adopted for better stability

Table 3.2
Final Design Vector, Sigma Vector, Constraints and Objective Function Value *for eighteen elements*

(D) Vector

4.25026	5.20700	32.1894	16.2139	5 66415	3.98030	9.44989
6.76165	4.12858	4.28736	4.28846	3.99824	3.96229	4.02469
3.97928	4.71694	4.04707	5.72798	5.01296	6.41219	5.30619
7.59059	6.35774	10.6066	8.85594	8.15849	5.75475	17.3116
9.78765	13.6371	12.4178	20.9039	15.0064	19.9580	14.1997
35.8934	30.0137	6 52259	14.4538	3 95233	71.2003	75.2055
15.3154	15.4204	12.4783				

Sigma Vector

15.4204	15.3154	14.4538	15.9511	32.1894	16.2139	17.3116
12.4783	10.6065	6.52259	9.44989	6 76165	5.66415	5.20700
4.25026	3.95233	3.98030	4.12858	71.2003	75.2055	63.5065
66.9593	49.4223	51.9055	36.0447	37.9901	26.4212	28.1964
18.0344	19.1751	13.2137	14.0551	10.2523	10.8998	8.22595
8.72824	5.37279	5.68797	3.97497	4 21044	3.03096	3.20653
2.28328	2.41456	1.55075	1.64379	1.08422	1.15228	0.71739
0.76838	0.34483	0.37549	0.10042	0.11206	0.00000	0.00000
4.28736	4.28846	3.99824	3.96229	4.02469	3.97928	4.71694
4.04707	5.72798	5.01296	6 41219	5.30619	7.59059	6.35774
10.6066	8.85594	8.15849	5.75475	11.9002	9.78765	13.6371
12.4178	20.9039	15 0064	19.9580	14.1997	35 8934	30.0137
16.1955	16.3270	15.5256	16.1955	15.4205	15.5256	15.4204
15.4205						

Interface Shear Equality Constraints

3.78042E-06	1 00136E-05	-2.61072E-05	-2.81334E-05	1.03166E-05	-9.53674E-07	2.23298E-06
7.62939E-06	6.54843E-06	1.28746E-05	-7.63589E-06	2.62260E-06	1 97856E-06	-4.17233E-06
1.00761E-05	-6.31809E-06	-6.79109E-06	2.50340E-06	-9.04665E-07	4.05312E-06	-1.13633E-06
-3.75509E-06	-3.87319E-07	-1.13249E-06	1.85883E-06	4.52995E-06	-2.01272E-06	-2.98023E-06
-1.06527E-06	1.19209E-07	5.41122E-07	1.19209E-07	-5.55798E-09	-1.49012E-07	

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Boundary Shear Equality Constraints

3.63831E-06	6.67572E-06	-6.67572E-06	-3 57628E-06	5.96046E-06	9.53674E-06	-2 86102E-06
4.76837E-06	1.43051E-06	3.81470E-06	-1 62530E-06	-1.34110E-06		

Non-linear No-yield Constraints (Inequality)

-1.61687E+02	-3.29590E-02	-5.02930E-02	-1.42879E+02	-5.02930E-02	-2.14006E+01	-3.03296E 00
-2.13948E+01	-1.67326E+02	-1.79947E+02	-1 67328E+02	-1.58709E+02	-8.78906E-03	-8.57813E 00
-8.90161E 00	-4.13330E-01	-2.44141E-02	-4.56653E 00	-6.65283E-03	-7.49512E-02	-2.72687E 00
-9.17023E 00	-1.48639E 00	-1.16593E+01	-6.77490E-03	-9.98444E-01	-5.34058E-03	-5.57388E-01
-1.68182E-01	-7.74551E-01	-4.63104E-03	-3.74603E-03	-2.29164E-01	-3.63007E-01	-5.28717E-03
-4.90135E-01	-2.49481E-03	-2.3[illegible]	-1.37718E-01	-8.44784E-02	-5.43594E-03	-8.22334E-02
-4.77600E-03	-2.71988E-03	-1.63841E-03	-1.28166E-01	-9.37843E-03	-3.49255E-02	-8.16898E-02
-4.59862E-03	-3.23677E-03	-4.76498E-01	-2.52533E-03	-1 70517E-03		

No-tension Constraints (Inequality)

-1.54204E+01	-1.53154E+01	-1.44538E+01	-1.59511E+01	-3.21894E+01	-1.62139E+01	-1.73116E+01
-1.24783E+01	-1.06065E+01	-6.52259E 00	-9.44989E 00	-6 76165E 00	-5.66415E 00	-5.20700E 00
-4.25026E 00	-3.95233E 00	-3.98030E 00	-4.12858E 00	-7.12003E+01	-7.52055E+01	-6.35065E+01
-6.69593E+01	-4.94223E+01	-5.19055E+01	-3.60447E+01	-3.79901E+01	-2.64212E+01	-2.81964E+01
-1.80344E+01	-1.91751E+01	-1.32137E+01	-1.40551E+01	-1.02523E+01	-1 [illegible]	-8.22595E 00
-8.72824E 00	-5.37279E 00	-5.68797E 00	-3.97497E 00	-4.21044E 00	-3.03096E 00	-3.20653E 00
-2.28328E 00	-2.41456E 00	-1.55075E 00	-1.64379E 00	-1.08422E 00	-1.15228E 00	-7.17398E-01
-7.68382E-01	-3.44834E-01	-3.75496E-01	-1.00424E-01	-1.12065E-01	-4.28736E 00	-4.28846E 00
-3.99824E 00	-3.96229E 00	-4.02469E 00	-3.97928E 00	-4.71694E 00	-4.04707E 00	-5.72798E 00
-5.01296E 00	-6.41219E 00	-5.30619E 00	-7.59059E 00	-6.35774E 00	-1.06066E+01	-8.85594E 00
-8.15849E 00	-5.75475E 00	-1.19002E+01	-9.78765E 00	-1.36371E+01	-1.24178E+01	-2 09039E+01
-1.50064E+01	-1.99580E+01	-1.41997E+01	-3.58934E+01	-3 00137E+01	-1.61955E+01	-1 63270E+01
-1.55256E+01	-1.61955E+01	-1.54205E+01	-1.55256E+01	-1.54204E+01	-1.54205E+01	

Optimal function value = 146.40

of the numerical scheme. However, due to non–availability of exclusive gradients Powell's conjugate direction method, a non-gradient based technique has still been retained in the Lysmer–Basudhar scheme and computations have been carried out. This adoption has worked very well in finding the solutions as has already been discussed, even with 127 number of design variables. However, there is a four-hundred fold increase in the number of function evaluations for fortyeight elements with that of six elements. It should be noted that this did not put any severe constraints in achieving the solutions as in all the cases these were obtained within 16 seconds of CPU time.

For eighteen elements, the final design vector, equality and inequality constraints along with the optimal value of the objective function are given in Table 3.2. The order of magnitude of the equality constraints is small enough to be considered equal to zero for all practical purposes. All the inequality constraints are negative showing that these are strictly satisfied.

Figs. 3.2(a) and 3.2(b) show the variation of the absolute value of the objective function with penalty parameter and the number of function evaluation respectively. From the figures it can be seen that the objective function attains a constant value when penalty parameter reaches a value of $10^{-3}$and the corresponding number of function evaluations is 73334. The steady nature of the objective function indicates a convergent solution.

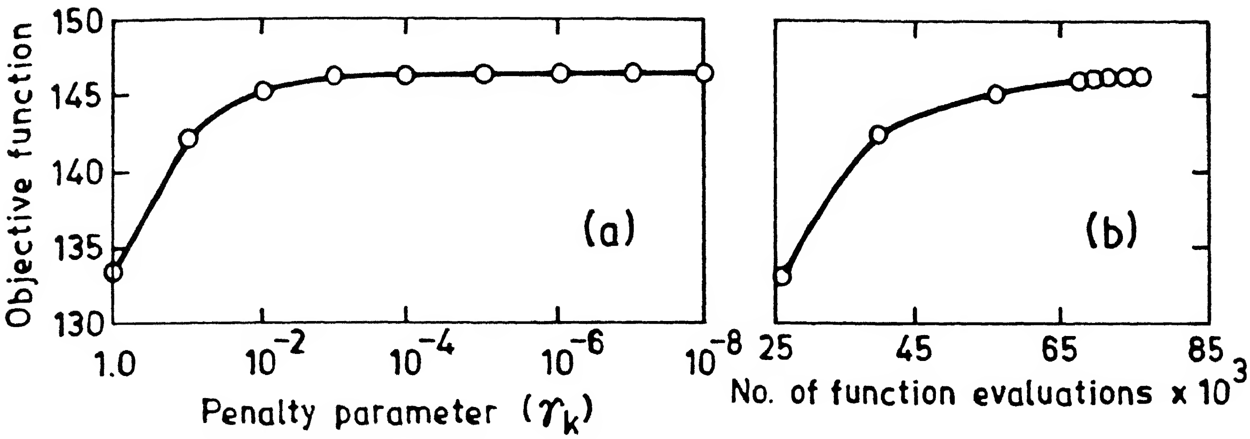

Fig 3.2 Variation of objective function with (a) Penalty parameter and (b)Number of function evaluations

As lower-bound analysis involves the generation of statically admissible stress field, it is of interest to study the state of stress in the soil medium corresponding to the optimal solution at the limiting state. The nearness of the state of stress at nodal point to the limiting state is judged by the stress strength ratio, defined as

$$\frac{(\sigma_z - \sigma_x)^2 + 4\tau_{zx}^2}{[(\sigma_z + \sigma_x)\sin\phi + 2C\cos\phi]^2}$$

where, σ_x and σ_z are the normal stresses on the plane through a nodal point in x and z direction respectively, τ_{zx} is the shear stress acting on the zx plane through a nodal point, C is the cohesion and ϕ is the angle of internal friction of the soil.

The complete stress field along with the stress strength ratio is shown in Table 3.3. It can be seen from the table that the obtained stress field is excellent as the stress-strength ratio at different nodal points for all the elements are very close to unity thus signifying the limiting equilibrium state.

Table 3.3
Stress Field and Stress-Strength Ratios at the Nodal Points for *eighteen elements*

Element No.	Nodal Point No.	σ_x	σ_z	τ_{zx}	Stress Strength ratio
1	1	15.4204	71.2003	0.0000	0.9506
1	2	15.4204	75.2055	0.0000	0.9999
1	3	15.4205	75.2055	0.0000	0.9999
2	1	15.3154	71.1834	0.0420	0.9562
2	3	15.4205	75.2055	0.0000	0.9999
2	4	15.5256	75.1887	0.0000	0.9940
3	1	14.4538	70.6321	0.7312	0.9990
3	4	15.5256	75.1887	0.0000	0.9940
3	5	16.1955	74.8230	0.0000	0.9535
4	1	15.9511	73.1624	1.2151	0.9479
4	5	16.1955	74.8231	0.0000	0.9535
4	6	16.2371	75.6737	0.0000	0.9568
5	1	12.9328	61.0891	4.8215	0.9999
5	6	14.0811	66.6897	4.4919	0.9962
5	7	13.3008	60.0142	6.6437	0.9962
6	1	11.6547	40.6404	9.9336	0.9996
6	7	12.4963	47.1428	9.8615	0.9999
6	8	12.3571	38.0969	11.0273	0.9960
7	1	11.5977	29.4715	10.7314	0.9999
7	8	12.3380	34.3410	11.2956	0.9999
7	9	12.3189	26.8534	11.1751	0.9961
8	1	11.5690	23.0060	10.3004	0.9837
8	9	12.3107	25.0217	11.0529	0.9977
8	10	12.2371	23.1742	10.7112	0.9802
9	1	11.1911	12.2577	8.2851	0.9991
9	10	11.9035	13.6866	8.9322	0.9969
9	11	11.6153	11.4019	8.1628	0.9999
10	1	10.1463	6.5225	5.8372	0.9962
10	11	11.0244	8.1584	6.7784	0.9991
10	12	10.2865	5.7454	5.4532	0.9944

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11	1	9.2873	4.2026	4.4225	0.9999
11	12	9.9722	4.9095	4.9378	0.9999
11	13	9.3521	3.7494	4.1065	0.9976
12	1	8.2344	2.5191	3.0935	0.9949
12	13	8.8583	2.9570	3.4809	0.9999
12	14	8.2551	2.2618	2.8439	0.9928
13	1	6.8313	1.1160	1.6904	0.9999
13	14	7.4100	1.4167	1.9988	0.9995
13	15	6.7402	0.9606	1.4557	0.9967
14	1	5.6190	0.4341	0.7811	0.9971
14	15	6.0652	0.5809	0.9494	0.9998
14	16	5.5468	0.3693	0.6270	0.9971
15	1	4.5396	0.0606	0.1462	0.9997
15	16	4.9581	0.1656	0.2807	0.9998
15	17	4.4038	0.0251	0.0086	0.9999
16	1	4.2531	0.0014	0.0160	0.9929
16	17	4.3484	0.0136	0.0165	0.9995
16	18	4.3044	0.0059	0.0243	0.9981
17	1	4.2787	0.0037	0.0237	0.9955
17	18	4.3218	0.0075	0.0190	0.9997
17	19	4.2870	0.0000	0.0252	0.9998
18	1	4.1285	0.0000	0.0000	0.9728
18	19	4.2873	0.0000	0.0252	0.9998
18	20	4.2884	0.0000	0.0000	0.9999

To study the extensibility of the stress field the original mesh for eighteen elements was extended as shown in Fig. 3.1. The new solution for this mesh was found to be 61.98 differing by only 0.70% from the previous one. Since the deviation is very marginal and insignificant for all practical purposes, the stress field may be considered to be extensible. Thus the obtained solution may be considered to be a true solution.

Since the analysis for the extended mesh is a general one, such study has not been repeated for meshes with other number of elements.

In Table 3.4, the obtained solutions are compared with the exact value and that from Bottero–Sloan's approach.

Table 3.4 Comparison of Bearing Capacity Solutions
Exact Value of N_q = 64.20

PRESENT SOLUTION			SLOAN'S SOLUTION			
No. of Elements	N_q	% Diff. from Exact Soln.	No. of Elements	$P\#$	N_q	% Diff from Exact Soln.
6	47.84	25.48		6	35.68	44.42
12	60.70	5.45		12	53.58	16.54
18	62.42	2.77	12	24	59.69	7.02
24	62.65	2.41		48	61.35	4.43
36	62.56	2.55		48*	61.11	4.81
48	61.91	3.56				
18*	61.98	3.45				

\# P = No. of Sides in linearized polygon
* Results for Extended mesh

Table 3.4 shows that the best solution from the present study differs by only 2.41% from the exact solution on the safer side whereas that of Sloan's differs from the same by 4.43%. The corresponding number of elements required to get the solution are 24 and 12 respectively. Sloan made a piece-wise linear approximation of the non-linear no-yield condition whereas for the Lysmer-Basudhar approach there was no necessity for such an approximation of the no-yield condition. However, with 12 elements the

method predicted a value which is marginally smaller than that predicted by Sloan. Taking eighteen elements instead of twelve elements a better solution (62.42) closer to the exact value (64.20) was obtained by Lysmer Basudhar approach. When the number of elements are twenty four there is a marginal increase in the N_q factor, beyond which further increase in number of elements infact reduces this factor. As such there is no need for consideration of number of elements more than twenty four for the estimation of the N_q factor. However, even eighteen elements would give excellent results.

From the table a direct comparison of the computational efficiency of the two approaches could not be made as the number of function evaluations to achieve the final optimal solution for twelve elements with Sloan's approach is not available. However, a qualitative and quantitative estimate can be made by comparing the final results obtained by these two approaches.

With twelve elements and forty eight sides of the linearized polygon Sloan's approach presented a better solution (61.35) than the one (60.7) obtained with the same number of elements and by using Lysmer-Basudhar approach ; the difference between these solutions is only 1.06%. But, when only twenty four sides of the linearized polygon are used Sloan's approach predicts a value (59.69) less than the value (60.7) obtained by the Lysmer-Basudhar approach differing by 1.69%. So by substantially increasing the number of sides better values can be predicted by the Sloan's approach than the Lysmer-Basudhar approach. However, the table also shows that just by increasing the number of elements from twelve to eighteen one predicts a better value (62.42) by the

Lysmer-Basudhar approach in contrast to the solution (61.35) obtained by using Sloan's approach with twelve elements and forty eight sides; the relative difference between these two solutions is 1.74% but the first one is closer to the exact one. It can be seen that Sloan's approach is strongly dependent on the number of sides of the linearized polygon simulating the no-yield condition. But Lysmer-Basudhar approach does not have any such drawbacks and as such, even with twelve elements a value of 60.7 could be obtained whereas Sloan's approach could not predict comparable values with less than twenty four sides of the linearized polygon.

3.3 Footing on Cohesive Soil

Another simple problem of surface strip footing on saturated fine grained soils under undrained condition has been chosen and is presented as follows.

3.3.1 The Problem

Fig. 3.3 shows a smooth strip surface footing resting on cohesive soil with $S_u = 1.0$ kPa. The objective is to determine the bearing capacity factor N_c for this footing.

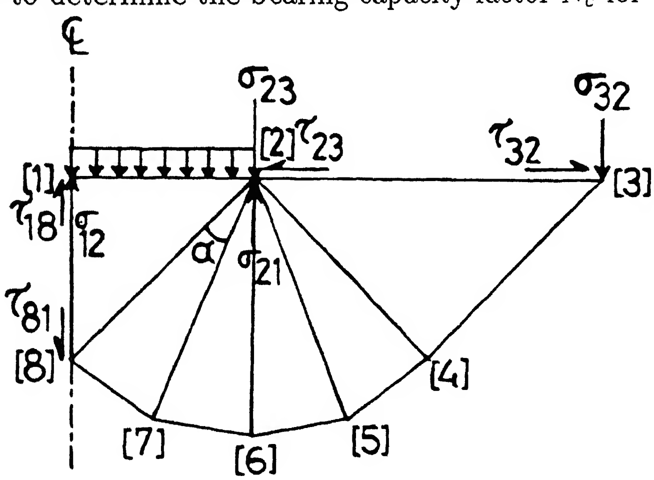

Fig 3.3 Strip footing on cohesive soil

3.3.2 The Objective Function

The objective function is $-(\sigma_{12} + \sigma_{21})$. Bearing capacity q_f is equal to half of the absolute value of the objective function.

3.3.3 The Boundary Conditions

The boundary conditions for the meshes shown in Fig. 3.3 are

$$\sigma_{23} = \sigma_{32} = 0 \text{ and}$$

$$\tau_{12} = \tau_{21} = \tau_{23} = \tau_{32} = \tau_{18} = \tau_{81} = 0$$

3.3.4 Results and Discussions

The adopted mesh geometry is shown in Fig. 3.3. The Bearing capacity factor N_c has been initially obtained for the mesh geometry (Fig. 3.3) with α equal to 22.5^0(6 elements). Subsequently the same was estimated with reduced value of $\alpha(10^0)$ in the radial shear zone and thus increasing the number of elements to 11. The mesh pattern is so chosen as to enable a direct comparison with the values obtained from the present approach with that of Munro–Chuang approach. For these values of $\alpha(22.5^0$ and $10^0)$ the corresponding lower bound values of N_c using the present approach are 4.98 and 4.96 in comparison to the upper bound solutions (Munro–Chuang approach) 5.18 and 5.15 respectively. The lower bound solutions obtained by Lysmer (1970), Arai and Tagyo (1985) and Sloan (1988) are 5.03, 5.04 and 5.08 respectively. The value 5.04 obtained by Arai and Tagyo is with twelve elements; he obtained a value of 4.67 when the elements were increased to twenty four. Sloan and Lysmer obtained the corresponding values

with eight and six elements respectively. Arai and Tagyo stated that the higher value of N_c with lesser number of elements is probably due to the over evaluation of the footing pressure caused by the rough discretization of stress field. Such studies were not conducted by Lysmer and Sloan whereas with both six and eleven elements, a convergent solution has been obtained from the presented approach.

3.4 Conclusions

The following generalized conclusions, based on the presented results and discussions, can be drawn:

- The stress field obtained by using the Lysmer-Basudhar approach has been found to be extensible and, as such, the predicted solution is a true lower bound.
- The obtained bearing capacity factors using Lysmer-Basudhar approach are closer to the exact solution than the same predicted by Bottero–Sloan's approach. The absolute errors of these two solutions from that of the exact solution (64.20) are 1.55 and 2.85 respectively and the corresponding relative errors are 2.41% and 4.43%. The adopted approach presented the best result very close to the exact solution for twenty four elements, but, further increase in the number of elements resulted in a marginal perturbation in the solution.
- Bottero–Sloan's approach of finding the lower bound solution is strongly dependent on the number of sides of the linearized polygon simulating the no–yield condition whereas Lysmer–Basudhar approach does not suffer from any such

drawbacks as it incorporates the non-linear no-yield constraints directly in the analysis.

- The bearing capacity factor N_c obtained by the present method for saturated fine grained soil under undrained condition is in close agreement with the values reported in literature. The percentage difference of the present solution from that of Lysmer (1970), Arai and Tagyo (1985) Sloan (1988) and Chuang (1992) are 1, 1.96, 1.19 and 2.9 respectively.

- Contrary to the general practice, retention of Powell's conjugate direction algorithm for unconstrained minimization in the Lysmer–Basudhar approach has been found to be prudent from the fact that it could handle large number of design variables without any problem.

CHAPTER 4

LOWER BOUND BEARING CAPACITY OF SURFACE STRIP FOOTINGS ON TWO LAYERED SOIL DEPOSITS

4.1 Introduction

The bearing capacity of homogeneous soils has been the subject of extensive study. But, in general, footings are to be located on natural stratified soil deposits exhibiting varying strength characteristics. A very common kind of such soil deposits is a soil layer of finite thickness overlying a thick stratum of another soil. The underlying stratum may either be a bed rock or another soil layer possessing different strength properties. Chapter 1 of the thesis gives a brief account of the available bearing capacity solutions for both homogeneous and stratified deposits. It is noticed that, as compared to the availability of bearing capacity solutions for homogeneous soil systems, the literature to predict the same for stratified deposits is less. The methods employed are based on the theory of limiting equilibrium (eg. Button, 1953; Mandel and Salencon 1972), finite element analysis (eg. Desai and Reese, 1970; Azam et al., 1991), experimental studies (eg. Tcheng, 1957; Brown and Meyerhof, 1969) lower bound limit analysis using the method of characteristics (Krishnamurthy, 1972; Reddy et al.,1989 and 1990), finite elements and non-linear programming (eg. Arai and Tagyo, 1985) and upper bound limit analysis (eg. Purushottamaraj et al.,1974; Reddy and Rao, 1983). As such, it is evident that apart from the application of the method of characteristics the only attempt to use other generalized methods to predict the lower bound limit load for such layered deposits has been made by Arai and Tagyo (1985). As the number of layers increases it

is very likely that the method of characteristics would be more difficult to use. As such, discrete elements and optimization based techniques being more flexible and general will be more appropriate for such problems. However, as already mentioned, only one such attempt has been made for a two layered soil deposits. So it is necessary to develop or extend other similar methods to such problems to asses their capability and validate the theoretical predictive model by comparing the obtained solutions with experimental observations as well as with other solutions reported in literature. With this in view one such method namely Lysmer-Basudhar approach (Lysmer, 1970; Basudhar, 1976; Basudhar et-al. 1979 and 1981) has been chosen and applied to study its suitability in solving such a problem and the same is presented as follows.

4.2 The Problem

Fig. 4.1 shows a strip footing of width B lying on the surface of a soil layer of thickness H having shear strength parameters C_1, ϕ_1 overlying another soil stratum having shear strength parameters C_2 and ϕ_2. The objective is to determine the bearing capacity of this footing for the different cases shown in Fig. 4.2.

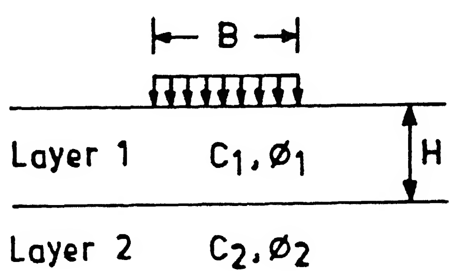

Fig 4.1 Details of Surface Strip Footing on Two Layered Soil Deposit

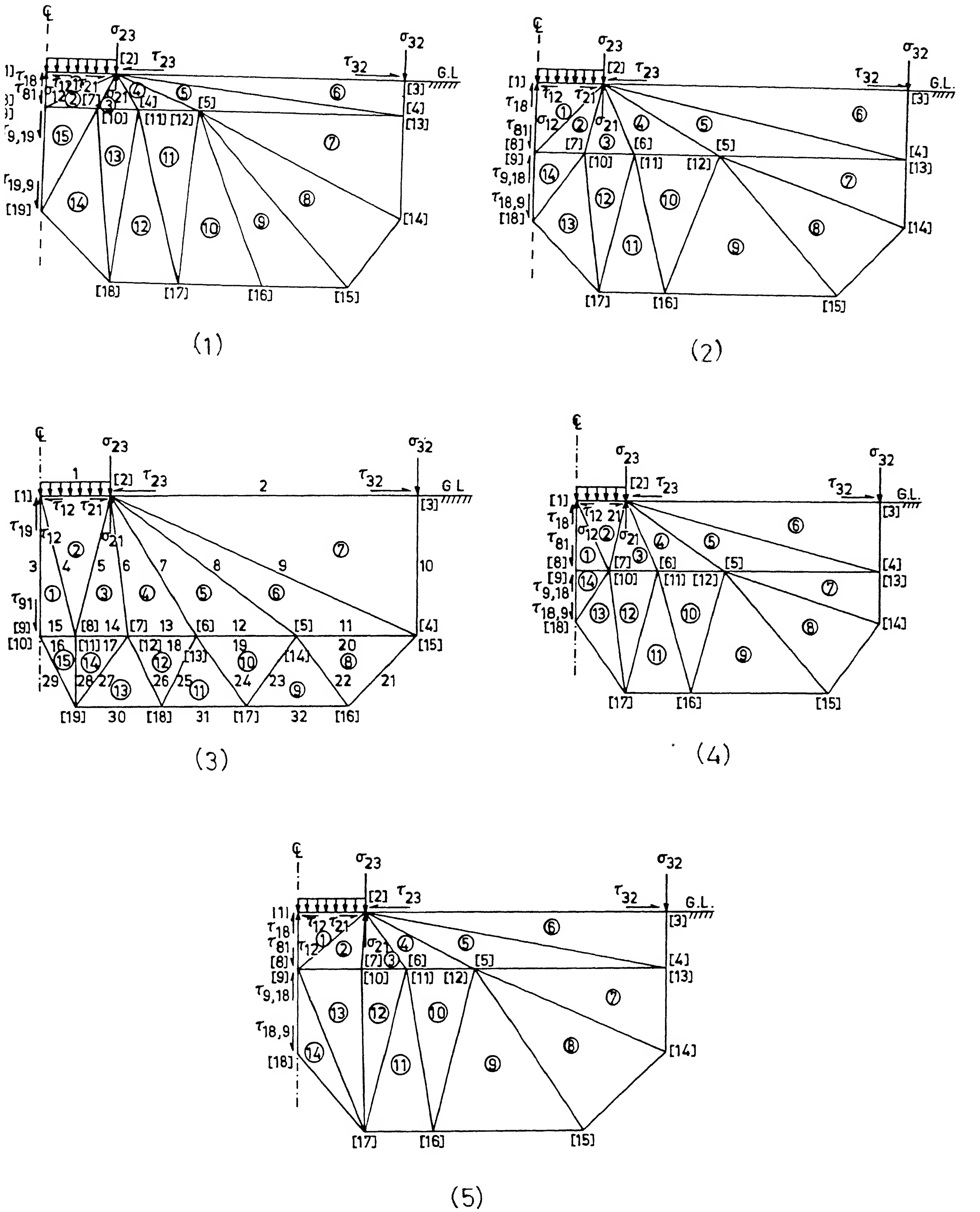

Fig. 4.2 Mesh patterns for footings on two layered soil deposits

The base of the foundation was considered to be rough for cases 1,2 and 3 (Brown and Meyerhof, 1969) and that for cases 4 (Arai and Tagyo, 1985) and 5 (Krishnamurthy, 1972) to be smooth. The zone under consideration is divided into a number of elements and the nodal points, elements and element sides are numbered in some arbitrary manner. The meshes with marked nodal point numbers, element numbers and element sides, used for analyses are shown in Figs. 4.2. It should be noted that the same element sides and nodes at the layer interface have been marked differently to take care of the different soil properties of the upper and lower layer. This also helps in taking care of the possibility of discontinuity of the stresses at the interface.

4.3 The Objective Function

The objective function is $-(\sigma_{12} + \sigma_{21})$ for all cases shown in Figs. 4.2. Bearing capacity q_f is equal to half of the absolute value of the objective function value.

4.4 The Boundary Conditions

The boundary conditions for the meshes shown in Figs. 4.2 are

$$\sigma_{23} = \sigma_{32} = 0 \qquad \text{(for all cases)}$$

for rough base(cases 1,2 and 3)

$$\tau_{23} = \tau_{32} = \tau_{18} = \tau_{81} = \tau_{9,18} = \tau_{18,9} = 0 \qquad \text{(for cases 1 and 2)}$$

$$\tau_{23} = \tau_{32} = \tau_{19} = \tau_{91} = 0 \qquad \text{(for case 3)}$$

$$\tau_{12} = C_1 + \sigma_{12} \tan\delta \text{ and } \tau_{21} = C_1 + \sigma_{21} \tan\delta \qquad \text{(for cases 1,2 and 3)}$$

for smooth base(cases 4 and 5)

$$\tau_{12} = \tau_{21} = \tau_{23} = \tau_{32} = \tau_{18} = \tau_{81} = \tau_{9,18} = \tau_{18,9} = 0$$

4.5 Results and Discussions

The lower bound bearing capacity solutions were obtained on CONVEX C-220 computer system. For saturated clay under undrained condition ($\phi_1 = \phi_2 = 0$) the computed value of the modified bearing capacity factor,N_{cm}, are compared with the experimental observations reported by Brown and Meyerhof (1969). For case 1($C_2/C_1 = 0.4$ and $H/B = 0.25$), the obtained modified bearing capacity factor N_{cm} is 2.45 which is 2.45% on the higher side of the experimental value 2.4 . Similarly for case 2($C_2/C_1 = 0.4, H/B = 0.5$) and case 3($C_2/C_1 = 0.2, H/B = 1.0$), the obtained values are 2.85 and 2.64 which, compared to the experimental values 2.8 and 2.5, are on the higher side by 1.7% and 4.3% respectively. Such a small difference in the values can be neglected as these are well within experimental errors. So it can be inferred that the Lysmer- Basudhar predictive model is excellent and the solutions can be treated as a true indication of the critical load for footing with rough base. For case 4($C_2/C_1 = 0.2, H/B = 2/3$) a better lower bound estimation of N_{cm} (1.72) is obtained from the present approach which is 1.71.69 reported by Arai and Tagyo (1985). However, the observed difference is more in case of footing with smooth base resting on a general cohesive frictional ($C-\phi$) layered soil deposit where for $C_2/C_1 = 0.4, \phi_2/\phi_1 = 1.25$(with $\phi_1 = 15^0$) and $H/B = 0.4$, the obtained modified bearing capacity factor N_{cm} is 5.95

which is 12.5% on the lower side of the value 6.8 predicted by Krishnamurthy (1972) using method of characteristics. For better appreciation these solutions are also presented in a tabular form (Table 4.1).

Table 4.1
Comparison of Bearing Capacity Factors for Footings on Two Layered Soil Deposits

Case	H/B	C_2/C_1	ϕ_2/ϕ_1	Brown and Meyerhof* (1969)	Krishna-murthy (1972)	Arai and Tagyo (1985)	Present Study	% Diff.
1	0.25	0.4	–	2.4	–	–	2.45	2.0
2	0.50	0.4	–	2.8	–	–	2.85	1.7
3	1.00	0.2	–	2.5	–	–	2.64	4.3
4	2/3	0.2	–	–	–	1.69	1.72	1.7
5	0.40	0.4	1.25	–	6.8	–	5.95	12.5

*Experimental Results

For case 3, the final design vector, equality and inequality constraints along with the objective function value at the optimum obtained starting from an arbitrarily chosen design vector are given in Table 4.2. The order of magnitude of the equality constraints is small enough to be considered equal to zero for all practical purposes. All the inequality constraints are negative showing that these are strictly satisfied.

Figs. 4.3(a) and 4.3(b) show the variation of the objective function with penalty parameter and the number of function evaluations respectively. From these figures it can be observed that the objective function attains a constant value when penalty

Table 4.2
Final Design Vector, Sigma Vector, Constraints and Objective Function Value *for eighteen elements*

(D) Vector

0.3430	0.5778	0.4416	0.0004	0.9771	0.5457	1.6303
1.6393	2.6093	2.6888	3.6068	3.0197	0.5399	0.4392
0.5098	0.3323	0.5720	0.3899	0.6498	0.6676	1.6109
0.7321	0.7868	2.7966	0.6779	0.8356	0.7141	0.2915
0.3642	0.6012	0.4370	0.3415	0.3881	0.3314	0.3631
0.1684	0.2886	0.4746	0.2840	2.4529	0.7378	0.1927
0.6447	1.5192	0.9003	2.4930			

Sigma Vector

2.4525	0.9909	1.5192	1.6109	0.3430	0.2396	0.0000
0.7378	0.5802	0.4746	0.4416	0.2346	0.3712	0.2429
0.1923	2.4930	2.7966	0.0000	0.0000	0.5346	1.6409
0.6474	1.8233	0.8702	1.1372	1.2213	0.6498	0.6096
0.2915	0.1230	0.1139	0.0111	0.0712	0.0000	0.5778
0.3333	0.0004	0.9771	0.5457	1.6303	1.6393	2.6093
2.6888	3.6068	3.0197	0.5399	0.4392	0.5098	0.3323
0.5720	0.3899	0.5176	0.6676	0.7870	0.7321	0.7868
0.8814	0.6779	0.8356	0.7141	0.5119	0.3642	0.6012
0.4370	0.3415	0.3881	0.3314	0.3631	0.1684	0.2886
0.3166	0.2840	0.2411	0.4321	0.1927	0.6447	0.4069
0.9003	0.3771					

Interface Shear Equality Constraints

-8.3446E-07	5.9604E-08	1.8284E-07	2.9802E-07	-1.7816E-07	0.0000E 00	2.4026E-08
0.0000E 00	2.6195E-07	-5.9604E-08	-2.9538E-08	-1.7027E-07	8.0049E-08	-5.9604E-08
2.1535E-08	1.4901E-07	3.5816E-08	-8.9407E-08	-8.4756E-09	5.9604E-08	-1.1561E-09
1.4901E-08	3.7090E-08	-8.9407E-08	-4.4703E-08	-5.9604E-08		

... contd. on next page

Boundary Shear Equality Constraints

2.9802E-08	-1.0658E-14	2.3841E-07	-1.7881E-07	-1.7816E-07	0.0000E 00

Constraints at Layer Interface

-2.0000E-01	-2.0000E-01	-2.0000E-01	-2.0000E-01	-2.0000E-01	-2.0000E-01	-2.0000E-01
-2.0000E-01	-2.0000E-01	-2.0000E-01				

Non-linear No-yield Constraints (Inequality)

-3.2143E-01	-2.0989E 00	-5.5018E-02	-3.1826E-03	-2.9897E-03	-7.0309E-03	-2.7489E 00
-7.9393E-04	-5.5069E-03	-4.7166E-03	-7.5731E-03	-1.8689E-01	-7.3600E-01	-4.0919E-01
-2.4981E-02	-8.7400E-01	-3.8828E 00	-8.1801E-03	-4.0000E 00	-3.9952E 00	-3.6624E 00
-1.3827E-01	-9.4530E-02	-1.2023E-01	-1.2584E-01	-1.1760E-01	-7.9354E-02	-1.2325E-01
-1.5131E-01	-9.4758E-02	-1.4884E-01	-1.5452E-01	-9.0286E-02	-9.1440E-02	-7.0488E-02
-1.5812E-01	-1.5921E-01	-1.3938E-01	-1.3638E-01	-4.2824E-02	-8.7521E-02	-1.4321E-01
-3.7421E-02	-6.6932E-02	-1.2258E-01				

No-tension Constraints (Inequality)

-2.4525E 00	-9.9092E-01	-1.5192E 00	-1.6109E 00	-3.4301E-01	-2.3962E-01	-7.3784E-01
-5.8021E-01	-4.7469E-01	-4.4160E-01	-2.3465E-01	-3.7122E-01	-2.4290E-01	-1.9233E-01
-2.4930E 00	-2.7966E 00	-5.3463E-01	-1.6409E 00	-6.4745E-01	-1.8233E 00	-8.7026E-01
-1.1372E 00	-1.2213E 00	-6.4985E-01	-6.0964E-01	-2.9154E-01	-1.2309E-01	-1.1395E-01
-1.1131E-02	-7.1299E-02	8.9923E-08	-5.7780E-01	-3.3331E-01	-4.1735E-04	-9.7715E-01
-5.4577E-01	-1.6303E 00	-1.6393E 00	-2.6093E 00	-2.6888E 00	-3.6068E 00	-3.0197E 00
-5.3999E-01	-4.3923E-01	-5.0986E-01	-3.3239E-01	-5.7203E-01	-3.8997E-01	-5.1762E-01
-6.6767E-01	-7.8705E-01	-7.3212E-01	-7.8689E-01	-8.8145E-01	-6.7790E-01	-8.3562E-01
-7.1410E-01	-5.1193E-01	-3.6427E-01	-6.0122E-01	-4.3700E-01	-3.4150E-01	-3.8811E-01
-3.3148E-01	-3.6314E-01	-1.6847E-01	-2.8864E-01	-3.1660E-01	-2.8408E-01	-2.4117E-01
-4.3211E-01	-1.9272E-01	-6.4473E-01	-4.0697E-01	-9.0035E-01	-3.7714E-01	

Optimal function value = 5.29

parameter reaches a value of 10^{-4} and the corresponding number of function evaluations is 46859. This steady nature of the objective function indicates a convergent solution.

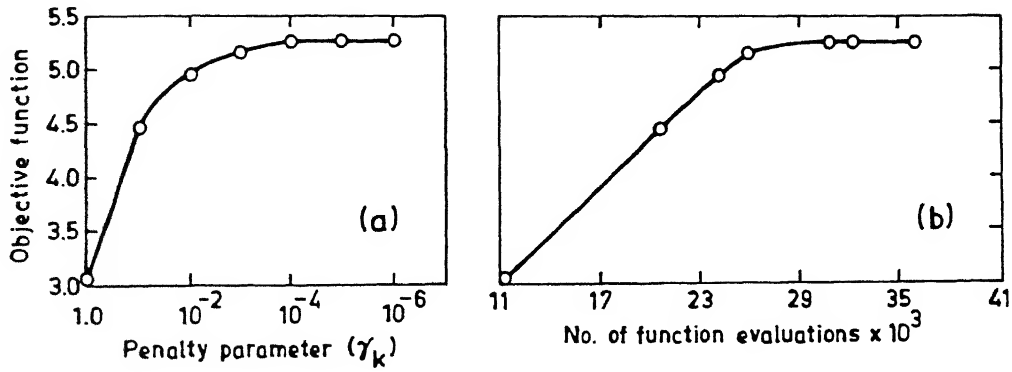

Fig 4.3 Variation of objective function with (a) Penalty parameter and (b)Number of function evaluations

The complete stress field along with the stress-strength ratio is shown in Table 4.3. It can be observed from the table that some of the nodal points are very close to limiting state. Similar stress fields are obtained for other cases, but for the sake of space and brevity they have not been presented herein.

Studies on extensibility have been undertaken for a typical case (case 3).The absolute value of the objective function for the original mesh [Fig. 4.2(c)] is 5.28. The nodal points 16,17,18 and 19 are extended downwards as shown in Fig. 4.4(a) and the value obtained for this case is 5.29 showing no remarkable increase from the previous value; then nodal points 3,4 and 15 are also extended along with the nodal points 16,17,18 and 19 ,Fig. 4.4(b), but the value 5.29 still remain unchanged.

Table 4.3
Stress Field and Stress-Strength Ratios at the Nodal Points *for case 3*

Element No.	Nodal Point No.	σ_x	σ_z	τ_{zx}	Stress Strength ratio
1	1	0.5346	2.4525	0.0000	0.9196
1	9	1.6409	3.0197	0.0000	0.4752
1	8	1.6409	3.6068	0.1417	0 9862
2	2	0.8139	2.7966	0.1282	0.9992
2	1	0.4983	2.4930	0.0674	0.9992
2	8	1.3999	2.7664	0.7289	0.9982
3	2	1.0876	1.5192	0.5159	0.3127
3	8	0.8460	2.6888	0.3883	0.9998
3	7	0.6116	2.6093	0.0303	0.9986
4	2	1.4644	1.6109	0 9967	0.9988
4	7	0.8646	1.6393	0.9209	0.9981
4	6	0.9006	1.6303	0.9056	0.9532
5	2	1.5525	0.3430	0.6710	0.8159
5	6	1.2832	0.5457	0.8727	0.8977
5	5	1.2053	0.9771	0.9903	0.9937
6	2	1.5439	0.2396	0.5968	0.7814
6	5	0.3425	0.0004	0.0050	0.0292
6	4	1.8005	0.3333	0.6780	0.9979
7	3	0.0000	0.0000	0.0000	0.0000
7	2	0.0687	0.0000	0.0000	0.0011
7	4	0.5778	0.0000	0.0302	0.0843
8	14	0.6444	0.7870	0 0186	0.1357
8	16	0.9640	0.7235	0.0435	0.4091
8	15	0.6544	0.7321	0.0918	0.2485
9	14	0.7611	0.5802	0.0188	0.2134
9	17	0.5829	0.3771	0.0031	0.2649
9	16	0.6722	0.9003	0.0846	0.5040
10	13	0.3642	0.5176	0.0575	0.2296
10	17	0.5585	0.5530	0.0465	0.0542
10	14	0.5648	0.6676	0.1169	0.4077
11	13	0.3898	0.4416	0.0046	0.0697
11	18	0.3365	0 4069	0 0114	0 0342
11	17	0.4462	0 6447	0.0870	0.4357
12	13	0.3222	0.3899	0 1264	0 4284
12	12	0.2909	0 5720	0 0512	0.5594
12	18	0 3447	0.3033	0.0062	0.0117
13	12	0.3788	0 3712	0.0134	0.0049
13	19	0.2526	0 1927	0.0652	0.1288
13	18	0.2846	0.4321	0.0216	0.1475
14	11	0.2886	0.5098	0 1306	0.7323
14	19	0.3166	0.2543	0.1309	0.4529
14	12	0.2889	0.3323	0.0610	0.1049
15	11	0.2886	0.4392	0.1580	0.7661
15	10	0.2367	0.5399	0.0166	0.5816
15	19	0 3166	0.1564	0.0542	0.2338

Finally two more elements 16 and 17 are added to the original mesh pattern of Fig. 4.2(c), the modified pattern is shown in Fig 4.4(c). The objective function value obtained for this case is 5.26 which is very marginally different from the values 5.28 and 5.29. To study the extensibility of this mesh pattern, the meshes have been extended as shown in Fig. 4.4(c) and the obtained objective function value is 5.28, again showing no appreciable difference from the value obtained with the unextended mesh pattern indicating that the stress field is extensible. Thus the obtained solution may be considered to be a true lower bound solution.

4.6 Conclusions

The following generalized conclusions, based on the presented results and discussions, can be drawn:

- Lysmer-Basudhar approach using discrete elements and non-linear programming can be used quite efficiently and reliably for predicting the lower bound bearing capacity of surface strip footings on a two layered soil deposit.
- The obtained lower bound bearing capacity factors for rough footings on two layered fine graind soil deposits under undrained condition ($\phi_u = 0$) are marginally higher $(1.7 - 4.3\%)$ than the available experimental values. The difference is well within permissible experimental error.
- For a general $C - \phi$ layered soil deposit, the obtained value of N_{cm} for smooth footing is 12.5% on the lower side of the value obtained using method of characteristics.
- Lysmer–Basudhar approach gives a better estimation of lower bound bearing capacity factor than that from Arai and Tagyo approach.

CHAPTER 5

GENERALIZED CONCLUSIONS AND SCOPE FOR FUTURE STUDIES

5.1 Conclusions

The generalized conclusions that are drawn based on the studies reported in chapters 3 and 4 are presented as follows:

- i) All the methods of isolating the optimal solution of bearing capacity problems based on linear programming (Lysmer, 1970; Sloan, 1988; Chuang, 1992) and non-linear programming (Basudhar, 1976; Arai and Tagyo, 1985) predicts values which are in close agreement for homogeneous soil deposits and the variations in the results are no more than 2%-3%.

- ii) Bottero-Sloan approach is considerably influenced by the number of sides of the linearized polygon simulating the no-yield condition whereas Lysmer-Basudhar does not suffer from any such drawbacks as it incorporates the non-linear no-yield constraints directly in the analysis.

- iii) Lysmer-Basudhar approach has been found to be quite efficient and reliable for predicting the lower bound bearing capacity of surface strip footing on a two layered soil deposit and predicts better solution than that by Arai and Tagyo.

  The predicted values using this method closely agree with the experimental observations. However, these solutions differs from that of method of characteristics solution by about 12.5%.

5.2 Scope for Future Studies

- i) As choice of the mesh pattern is very important and at present is mostly guided by intution, previous experience and experimental observations, more studies are needed to provide guidelines for proper discretization of the medium.

- ii) As the adopted method of analysis does not provide any information regarding the displacement, appropriate constitutive relationship for the soil be included in the model and a mixed formulation be made. This would result in obtaining an exact solution.

- iii) Extension of Lysmer–Basudhar approach to reinforced soils simulating the presence of the soil and the reinforcing elements separately than treating the whole soil–reinforced composite mass as an equivalent homogeneous anisotropic material.

- iv) Design charts based on lower bound solutions for unconventional problems in soil mechanics like bearing capacity of footings located above voids may be prepared.

APPENDIX

PROOF THAT IT IS SUFFICIENT TO SATISFY THE NO-YIELD CONDITION AT THE CORNERS OF THE ELEMENTS

It has been proved by Lysmer (1993) that, if the no-yield condition is satisfied at the corners of an element, no points within the element will be above yield. The same is reproduced here as follows.

The no-yield condition is:

$$(\sigma_z - \sigma_x)^2 + (2\tau_{zx})^2 < [(\sigma_z + \sigma_x)\sin\phi + 2C\cos\phi]^2 \tag{1}$$

or, it can be written as

$$A^2 + B^2 < C^2 \tag{2}$$

where A, B and C are generalized stresses :

$$\begin{aligned} A =& \quad \sigma_z - \sigma_x \\ B =& \quad 2\tau_{zx} \\ C =& \quad (\sigma_z + \sigma_x)\sin\phi + 2C\cos\phi \end{aligned} \tag{3}$$

These stresses vary linearly over the zx-plane since σ_z, σ_x and τ_{zx} have this property. Also $C \geq 0$, since a no-tension stress field is assumed. Let us now consider two points, 0 and 1, at which the linear stress field. For these points we have by equation (2),

$$\begin{aligned} &point0 \quad r_0^2(A_0^2 + B_0^2) = C_0^2 \quad C_0 \geq 0 \\ &point1 \quad r_1^2(A_1^2 + B_1^2) = C_1^2 \quad C_1 \geq 0 \end{aligned} \tag{4}$$

where $r_0 > 1$ and $r_1 > 1$ are known constants.

The generalized stresses at points on a line between points 0 and 1 vary linearly. Thus

$$A(t) = (1-t)A_0 + tA_1$$

$$B(t) = (1-t)B_0 + tB_1 \tag{5}$$

$$C(t) = (1-t)C_0 + tC_1$$

where $0 \leq t \leq$ is a parameter which has the values 0 and 1 at points 0 and 1 respectively.

Substuting equation (4) into the expression for $C(t)$ we get,

$$C(t) = (1-t)r_0\sqrt{A_0^2 + B_0^2} + tr_1\sqrt{A_1^2 + B_1^2} \tag{6}$$

Thus the right hand side of the equation (2) is:

$$\begin{aligned} C^2(t) = \ & (1-t)^2 r_0^2 \ (A_0^2 + B_0^2) + t^2 r_1^2 (A_1^2 + B_1^2) \\ & +2t(1-t)r_0 r_1 \ \sqrt{(A_0^2 + B_0^2)(A_1^2 + B_1^2)} \end{aligned} \tag{7}$$

which for points between points 0 and point 1 satisfy

$$\begin{aligned} C^2(t) > \ & (1-t)^2 \ (A_0^2 + B_0^2) + t^2 (A_1^2 + B_1^2) \\ & +2t(1-t) \ \sqrt{(A_0^2 + B_0^2)(A_1^2 + B_1^2)} \end{aligned} \tag{8}$$

The left hand side of (2) is, by (5),

$$A^2(t) + B^2(t) = (1-t)^2(A_0^2 + B_0^2) + t^2(A_1^2 + B_1^2) + 2t(1-t)(A_0A_1 + B_0B_1) \tag{9}$$

In the range $0 \leq t \leq 1$, we have $2t(1-t) \geq 0$. Thus by comparison between (8) and (9), we conclude that the no-yield condition

$$A^2(t) + B^2(t) < C^2(t) \tag{10}$$

is satisfied in the range $0 \leq t \leq 1$, provided it is true that

$$A_0A_1 + B_0B_1 \leq \sqrt{(A_0^2 + B_0^2)(A_1^2 + B_1^2)} \tag{11}$$

This is indeed so, which can be seen from the following:

$$(A_0B_1 - B_0A_1)^2 \geq 0 \qquad \text{(self evident)}$$

$$(A_0B_1)^2 + (B_0A_1)^2 \geq 2A_0B_1B_0A_1$$

Adding $(A_0A_1)^2 + (B_0B_1)^2$ to both sides, we get

$$(A_0^2 + B_0^2)(A_1^2 + B_1^2) \geq (A_0A_1 + B_0B_1)^2$$

$$\Leftrightarrow \quad \sqrt{(A_0^2 + B_0^2)(A_1^2 + B_1^2)} \geq |A_0A_1 + B_0B_1|$$

which confirms that (11) and thus (10) is satisfied. But this means that, *in a linear stress field, all points on a line between two no-yielding points will be non-yielding*

This theorem guarantees that, if the no-yield condition is satisfied at the corners of an element, no points within the element will be above yield.

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